Encapsulating quantifiers with the typed variables

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Abstract: The asymmetry in the translation of Natural language sentences involving existential and universal quantifiers is well known. It is possible to get rid of this asymmetry by postulating 'quantified typed' variables. In this presentation, we define a 'quantified typed' variable and the algebra associated with these variables to prove the deductions using the method of reductio-ad-absurdum.

The translation of quantifiers into Predicate Logic is asymmetric.

Every Man is mortal.
is translated into Predicate Logic as
$\forall x[Man(x) \rightarrow Mortal(x)]$

while

Some man is intelligent.
is translated as
$\exists x[Man(x).Intelligent(x)]$

Further $\rightarrow$ is purely a material implication.

i.e.

$\forall x[Man(x) \rightarrow Mortal(x)]$

is truth functionally equivalent to

$\forall x[\overline{Man}(x) + Mortal(x)]$

Thus there is an asymmetry in the way predicates with universal and existential quantifiers are expressed in Logic.

Every Man is mortal.
is represented as
$\forall x[\overline{Man}(x) + Mortal(x)]$

while

Some man is intelligent.
is represented as
$\exists x[Man(x).Intelligent(x)]$

If the variables are typed, this asymmetry disappears.

Every Man is mortal.

$\forall x \in man[Mortal(x)]$

Some man is intelligent.

$\exists x \in man[Intelligent(x)]$

But still the quantification and the variables are separated from each other.

- the scope of quantifiers is marked explicitly using ‘parenthesis’,
- the dependency is indicated by the order of the quantifiers.

Linear representation is not suitable for representing branching quantifiers.

Some relative of each villager and some relative of every townsman hate each other.

$$\forall x \exists y [villageman(x).relative(y, x)]$$

hate(y,v)

$$\forall u \exists v [townsman(u).relative(v, u)]$$

Proposal:
Package the 'quantification' and 'type' together.

Use Vowels for variables with Existential quantification.
Use Consonants for variables with Universal quantification.

All men are mortal.
mortal(k:man)

Some man is intelligent.
intelligent(a:man)

We write this succinctly as mortal(k) and intelligent(a), if the types of the variables ‘k’ and ‘a’ have already been declared earlier.
Thus there is an uniformity in the representation.

What about the dependency and scope?

$\forall x \exists y F(x, y) \Longrightarrow F(k, k \rightarrow a)$
where $k \rightarrow a$ denotes the existential quantifier 'a'
depends on the universal quantifier 'k'.

whereas
$\exists y \forall x F(x, y) \Longrightarrow F(k, a)$.

Variables in different scopes are represented by different letters.
Thus
$\exists y\{[\forall x F(x,y)][\forall x G(x,y)]\}$
is represented as
$F(k,a).G(g,a)$

Use of same variables in different predicates indicate co-reference.

Why branchiing quantifiers is not a problem?

Some relative of each villager and some relative of every townsman hate each other.

$villageman(k).relative(k \rightarrow a, k).$
$townsman(g).relative(g \rightarrow i, g).$
$hate(k \rightarrow a, g \rightarrow i)$

Algebra of Predicates with Quantified typed variables

Let ‘a’, and ‘e’ be the existential quantifiers and
‘k’, and ‘g’ be the universal quantifiers over the domain M.

Then,

$P(k).\bar{P}(k) = 0$

$P(k).\bar{P}(a) = 0$

$P(k).\bar{P}(g) = 0$

$P(a).\bar{P}(a) = 0$

$P(a).\bar{P}(e)\ != 0$

$$P(k) + \bar{P}(k) = 1$$
$$P(k) + \bar{P}(a) = 1$$
$$P(k) + \bar{P}(g) = 1$$
$$P(a) + \bar{P}(a) = 1$$
$$P(a) + \bar{P}(e) \mathrel{!=} 1$$

$$\overline{P(a)} = \bar{P}(k)$$
$$\overline{P(k)} = \bar{P}(a)$$

Algebra of Predicates with more than one arguments

Let for each i, a_i and b_i be of same type. Then
$P(a_1, a_2, ..., a_n).\bar{P}(b_1, b_2, ..., b_n) = 0$
and
$P(a_1, a_2, ..., a_n) + \bar{P}(b_1, b_2, ..., b_n) = 1$
if $\forall i$, one of the a_i or b_i is universal,
or $a_i = b_i$,
or either a_i or b_i is existential.

In case of existential quantifiers, it is immaterial whether they are dependant on other universal quantifiers or not.

Or, in other words,

$P(a_1, a_2, ..., a_n).\bar{P}(b_1, b_2, ..., b_n)! = 0$
and
$P(a_1, a_2, ..., a_n) + \bar{P}(b_1, b_2, ..., b_n)! = 1$

if and only if for some i, $a_i! = b_i$, and both a_i and b_i are existential.

Negation of a quantified typed predicate

Let $P(a_1, a_2, ..., a_{m+n})$ be a predicate.
Let 'm' of these arguments be universal and 'n' of them be existential.
Let these be denoted by $c_1, c_2, ..., c_m$ and $d_1, d_2, ..., d_n$ respectively.
Let C_{m*n} be a matrix with C[i,j] = 1 if c_j depends on c_i, and else 0.

Let D = complement(C), where complement(C) is obtained by taking the binary complement of each element of the transposed matrix, and changing the names of the rows from consonantes to vowels and vowels to consonants.

$$\overline{P(a_1, a_2, ..., a_{m+n})} = \bar{P}(b_1, b_2, ..., b_{m+n})$$

where

b_i is existential if a_i is universal, and

b_i is universal if a_i is existential.

Further, in case of existential quantifiers, the i^{th} existential variable depends on j^{th} universal variable, if D[i,j] = 1.

Negation of $P(k, a, k \rightarrow i, g, \{k, g\} \rightarrow e)$:

The dependency matrix D for the predicate P is

	a	i	e
k	0	1	1
g	0	0	1

The complement(D) is

	c	j	t
o	1	0	0
u	1	1	0

Hence the negation of P $\Rightarrow \bar{P}(c \rightarrow o, c, j, \{c, j\} \rightarrow u, t)$

Every child eats a banana

is represented as

$eat(k : child, k \rightarrow a : banana)$

One who eats a banana

is represented as

$eat(*, * \rightarrow a : banana)$

One who does not eat a banana

is represented as

$\overline{eat}(*, k : banana)$

Every person who eats a banana gains weight

is represented as

gains_weight(k:{person, P1})

where P1 = $eat(*, * \rightarrow a : banana)$

There exists a person who eats a banana and still does not gain weight

is represented as

$\overline{gains\_weight}(a : \{person, P1\})$

where P1 = $eat(*, * \rightarrow a : banana)$

If P1 is any property, then

$$P(a : P_1) = P_1(a).P(a)$$

$$P(k : P_1) = \overline{P_1}(k) + P(k)$$

There is a professor who is liked by every student who likes at least one professor.

Every student likes some professor or the other.

Therefore, there is a professor who is liked by all students.

There is a professor who is liked by every student who likes at least one professor.

like(k:{stud, P_1},i:prof)
where $P_1 = like(*, * \rightarrow a : prof)$

Or we may express it as
$like(k : \{stud, like(*, * \rightarrow a : prof)\}, i : prof)$

Every student likes some professor or the other.

$like(g : stud, g \rightarrow u : prof)$

There is a professor who is liked by all students.

like(c:student,e:prof)

$like(k : \{stud, like(*, * \rightarrow a : prof)\}, i : prof)$– (1)

$like(g : stud, g \rightarrow u : prof)$ – (2)

———-

like(c:stud,e:prof) – (3)

Proof by Contradiction

negation of (3) is:

$\overline{like}(t \rightarrow o : stud, t : prof)$ – (3')

Let t=i (clue from 1).

(3') reduces to $\overline{like}(i \rightarrow o : stud, i : prof)$ – (4)

Let $g = i \rightarrow o$ (or simply o) in (2).
$like(o : stud, o \rightarrow u : prof)$ – (5)

Let $k = i \rightarrow o$ (or simply o).

So (1) reduces to

$like(o : \{stud, like(*, * \rightarrow a : prof)\}, i : prof)$– (6)

Thus (6) contradicts with (4).

Hence the proof.

Another way of proving this is just by the method of algebraic simplification. This has an advantage over the previous one, since now there is no problem of instantiation.

In what follows, we replace the predicate 'like' by 'L', and also do not mention the domains.

$[L(k, i) + \bar{L}(k, k \rightarrow a)]$ — (1")

$L(g, g \rightarrow u)$ — (2")

$\bar{L}(t \rightarrow o, t)$ — (3")

By taking the product of all these 3, and applying distributivity, we get

$$L(k,i).L(g,g \rightarrow u).\bar{L}(t \rightarrow o,t) +$$
$$\bar{L}(k,k \rightarrow a).L(g,g \rightarrow u).\bar{L}(t \rightarrow o,t)$$
$$= 0 + 0 \text{ (refer slide 15)}$$
$$= 0$$