Vargaprakṛti

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Vargaprakṛti: Some terminology

Vargaprakṛti (equation of a **square** with some given **co-efficient**) is the name given to the equations of type $Nx^2 + C = y^2$ which is a general form of Pell's equation $Nx^2 + 1 = y^2$, where N is is a positive nonsquare integer.

Vargaprakṛti: Some terminology

iṣṭaṁ hṛsvaṁ tasya vargaḥ prakṛtyāḥ
kṣuṇṇau yukto varjitā vā saḥ yena
mūlaṁ dadyāt kṣepakaṁ tam dhanarṇe
mūlaṁ tat ca jyeṣṭa mūlaṁ vadanti

Assume a number (x), which is called hrasva or kaniṣṭa, its square should be multiplied by prakṛti and add or subtract a suitable number, called kṣepaka, from it to make the remainder a perfect square. The square root of this number is called a larger root or jyeṣṭha mūlam.

Vargaprakṛti: Some terminology

In the equation $Nx^2 + C = y^2$

- N: Prakṛti
- C: kṣepaka
- x: Kaniṣṭha root (smaller root)
- y: Jyeṣṭha root (larger root)

Bhāvanā

Bhāvanā is the process of obtaining several solutions from a given solution of a vargaprakṛti equation. Assuming that there is one solution, below Bhaskaracharya describes various methods of obtaining other solutions derived from this basic solution.

hrasva jyeṣṭa kṣepakān nyasya teṣām

For the equation of the form $Nx^2 + C = y^2$, write down suitable values of x, y and c in order.

tān anya-anya-adaḥ niveśya krameṇa

Below them write the values of x,y and c of other similar(or the same) equation (with different C) in the same order.

sādhya-anyebhyaḥ bhāvanābhiḥ bahūni

By the process of bhāvanā, we can obtain many values of x and y.

mūlāni eṣām bhāvanā procchate ataḥ

The process is called Bhāvanā.

Vargaprakṛti: Rule 1a

vajra-abhyāsau jyeṣṭa laghvoḥ tadaikyaṁ hṛsvaṁ

The sum of the cross-products of smaller and greater roots will be the smaller root of the new equation.

laghu ārohatiḥ ca prakṛtyā kṣuṇṇā
jyeṣṭa abhyāsayuk jyeṣṭamūlaṁ

The products of smaller roots multiplied by the prakṛti added to the product of the greater roots will be the greater root of the new equation.

tatrābhyāsaḥ kṣepayoḥ kṣepakaḥ syāt
The product of the constants will be the constant of the new equation.

Vargaprakṛti: Rule 1a

If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + k = y^2$
and $(x, y) = (\alpha_1, \beta_1)$ is a solution for the equation $Nx^2 + k_1 = y^2$
then the solution of $Nx^2 + kk_1 = y^2$ is given by
$(x, y) = (\alpha\beta_1 + \alpha_1\beta, N\alpha\alpha_1 + \beta\beta_1)$

Vargaprakṛti: Rule 1b

hrasvaṁ vajrābhyāsayoḥ antaram vā

Otherwise, the difference of the cross-products of smaller and greater roots will be the new smaller root

laghvoḥ ghātaḥ yaḥ prakṛtyā vinighnaḥ

and the products of smaller roots multiplied with prakriti

ghātaḥ yaḥ ca jyeṣṭayoḥ tad viyogaḥ jyeṣṭaṁ

reduced by the product of greater roots is the new greater root.

kṣepaḥ atra api ca kṣepaghātaḥ

Here also the product of the constants will be the constant of the new equation.

Vargaprakṛti: Rule 1b

If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + k = y^2$
and $(x, y) = (\alpha_1, \beta_1)$ is a solution for the equation $Nx^2 + k_1 = y^2$
then the solution of $Nx^2 + kk_1 = y^2$ is given by
$(x, y) = (\alpha\beta_1 - \alpha_1\beta, N\alpha\alpha_1 - \beta\beta_1)$

Vargaprakṛti: Rule 1: Proof

Let $Nx^2 + k = y^2$, and
$N_1x_1^2 + k_1 = y_1^2$.

Hence,
$kk_1 = (y^2 - Nx^2)(y_1^2 - N_1x_1^2)$
$kk_1 = (y^2y_1^2 + N^2x^2x_1^2 - Nx^2y_1^2 - Nx_1^2y^2)$
$Nx^2y_1^2 + Nx_1^2y^2 + kk_1 = y^2y_1^2 + Nx^2Nx_1^2$
$Nx^2y_1^2 + Nx_1^2y^2 \pm 2Nxx_1yy_1 + kk_1 = y^2y_1^2 + N^2x^2x_1^2 \pm 2Nxx_1yy_1$
$N(xy_1 \pm x_1y)^2 + kk_1 = (yy_1 \pm Nxx_1)^2$

which is of the form

$NX^2 + K = Y^2$, where
$X = xy_1 \pm x_1y,\ Y = yy_1 \pm Nxx_1, and, K = kk_1$

hence the solution of $Nx^2 + kk_1 = y^2$ is given by
$(x, y) = (\alpha\beta_1 \pm \alpha_1\beta, N\alpha\alpha_1 \pm \beta\beta_1)$

Vargaprakṛti: Rule 1:Special case

If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + k = y^2$ then $Nx^2 + k^2 = y^2$ will have the solution $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + 1 = y^2$ then another solution is give by $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

Vargaprakṛti: Rule 1 Special Case :Proof

$Nx^2 + 1 = y^2$
$1 = (y^2 - Nx^2)^2$
$1 = y^4 + N^2x^4 - 2Nx^2y^2$
$1 + 4Nx^2y^2 = y^4 + N^2x^4 + 2Nx^2y^2$
$1 + N(2xy)^2 = (y^2 + Nx^2)^2$
Let $2xy = X$ and $y^2 + Nx^2 = Y$
Then (X, Y) is a solution of $NX^2 + 1 = Y^2$
Thus if (α, β) is a solution of $1 + Nx^2 = y^2$, then
$(X, Y) = (2\alpha\beta, N\alpha^2 + \beta^2)$ is also its another solution.

Vargaprakṛti: Rule 2

iṣṭa varga hṛtaḥ kṣepaḥ
kṣepasya ādiṣṭa bhājite
mūle te staḥ athavā kṣepaḥ
kṣuṇṇaḥ kṣuṇṇe tadā pade

If the constant is divided by the square of a desired number to form new constant, the values of x and y should be divided by the same desired number to get the new values of x and y.

Similarly, if the constant is multipied by the square of a desired number, the values of x and y are to be multipled by the same desired number to get the new values of x and y.

Vargaprakṛti: Rule 2

If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + c = y^2$ then the solution of $Nx^2 + ck^2 = y^2$ is given by $(x, y) = (\alpha * k, \beta * k)$

If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + c = y^2$ then the solution of $Nx^2 + \frac{c}{k^2} = y^2$ is given by $(x, y) = (\frac{\alpha}{k}, \frac{\beta}{k})$

Vargaprakṛti: Rule 3

How to get the initial solution?

iṣṭa varga prakṛtyoḥ
yad vivaraṃ tena vā bhajed
dvighnamiṣṭaṁ kaniṣṭaṁ
tat padam syād eka saṁyutau
tataḥ jyeṣṭam

If the constant (c) is equal to 1, the value of x can be obtained by dividing twice a desired number by the difference between the prakṛti and the square of this desired number. From this the greater root (y) can be obtained.

So $x = \frac{2a}{N-a^2}$ is a possible value of the smaller root for an equation $Nx^2 + 1 = y^2$ where a be some desired number.

iha anantyam bhāvanā tatheṣataḥ

By Bhāvanā, a number of such solutions can be obtained.

Vargaprakṛti: Rule 3 Proof

Consider the equation $Nx^2 + 1 = y^2$.
let a be some desired number.
Then, $(N - a^2)^2 + 4Na^2 = (N + a^2)^2$

Dividing both sides by $(N - a^2)^2$, we get
$1 + \frac{4Na^2}{(N-a^2)^2} = \frac{(N+a^2)^2}{(N-a^2)^2}$
i.e., $1 + N(\frac{2a}{N-a^2})^2 = \frac{(N+a^2)^2}{(N-a^2)^2}$

So $x = \frac{2a}{N-a^2}$ is a possible value of the smaller root.

An Example

ko varga aṣṭa hataḥ sa-ekaḥ
kṛtiḥ syād gaṇaka ucyatām
ekādaśa guṇaḥ ko vā
vargaḥ sa-ekaḥ kṛtibhavet

O, Mathematician, tell me the number whose square multiplied by 8 and increased by one becomes a square number.

Also tell me the number whose square multiplied by 11 and increased by one gives a square number.

Solve the eqautions: $8x^2 + 1 = y^2$, and $11x^2 + 1 = y^2$.

An Example: Solution

Consider the eqaution: $8x^2 + 1 = y^2$

For some a, $x = \frac{2*a}{8-a^2}$.
For $a = 1$, $x = \frac{2}{7}$
For $a = 2$, $x = \frac{4}{4} = 1$

For $x = 1$, $y = 3$. If $(1, 3)$ and $(1, 3)$ are two roots, then using bhāvanā, we get
$(x, y) = (1 * 3 + 1 * 3, 8 * 1 * 1 + 3 * 3) = (6, 17)$ as the second solution.

If $(1, 3)$ and $(6, 17)$ are two roots, then using bhāvanā, we get
$(x, y) = (1 * 17 + 3 * 6, 8 * 1 * 6 + 3 * 17) = (35, 99)$ as the third solution.

An Example: Solution

Consider the eqaution: $11x^2 + 1 = y^2$

For some a, $x = \frac{2*a}{11-a^2}$.
For $a = 1$, $x = \frac{2}{10}$
For $a = 2$, $x = \frac{4}{7}$
For $a = 3$, $x = \frac{6}{2} = 3$

For $x = 3$, $y = 10$. If $(3, 10)$ and $(3, 10)$ are two roots, then using bhāvanā, we get
$(x, y) = (2 * 3 * 10, 11 * 3 * 3 + 10 * 10) = (60, 199)$ as the second solution.

If $(3, 10)$ and $(60, 199)$ are two roots, then using bhāvanā, we get
$(x, y) = (3 * 199 + 10 * 60, 11 * 3 * 60 + 10 * 199) = (1197, 3970)$
as the third solution.

An Example: Alternative method

Consider the eqaution: $11x^2 + 1 = y^2$

Find some x such that for some k, $11x^2 + k$ is a perfect square.
Let $x = 1$ and $k = -2$ is such pair, since $11 * 1^2 - 2 = 9 = 3^2$.
Thus $(1, 3)$ is a solution to $11x^2 - 2 = y^2$.

Using Bhāvanā,
If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + k = y^2$
then $Nx^2 + k^2 = y^2$ will have the solution
$(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

Hence $(2 * 1 * 3, 11 * 1^2 + 3^2) = (6, 20)$ is a solution of
$11x^2 + 4 = y^2$.

Using Bhāvanā again,
If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + c = y^2$
then the solution of $Nx^2 + \frac{c}{k^2} = y^2$ is given by
$(x, y) = (\frac{\alpha}{k}, \frac{\beta}{k})$

$(\frac{6}{2}, \frac{20}{2}) = (3, 10)$ is the solution of $11x^2 + 1 = y^2$.

example

ko vargaḥ śadguṇaḥ tryādhyo
dvādaśādhyo athavā kṛtiḥ
yuto vā pañca saptatyā
triśatyā vā kṛtir bhavet

Which is that number whose square multiplied by 6 and increased by 3, 12, 75 and 300 respectively becomes square numbers.

$$6x^2 + 3 = y^2$$
$$6x^2 + 12 = y^2$$
$$6x^2 + 75 = y^2$$
$$6x^2 + 300 = y^2$$

ko vargaḥ śadguṇaḥ tryādhyo
dvādaśādhyo athavā kṛtiḥ
yuto vā pañca saptatyā
triśatyā vā kṛtir bhavet

Which is that number whose square multiplied by 6 and increased by 3, 12, 75 and 300 respectively becomes square numbers.

$6x^2 + 3 = y^2$; Solution $(1, 3)$
$6x^2 + 12 = y^2$; Solution $(2, 6)$
$6x^2 + 75 = y^2$; Solution $(5, 15)$
$6x^2 + 300 = y^2$; Solution $(10, 30)$

Rule

varga cchinne guṇe hrasvam tat padena vibhājayet.

If the guṇa (prakṛti) is divided by the square of any number, then the smaller root of the reduced equation should be divided by the square root of the square used for division, to get the correct root of the original euqation.

For example, in the equation $8x^2 + 1 = y^2$, the prakṛti 8 can be divided by 2^2 to simplify the given equation as $2x^2 + 1 = y^2$. Root of this reduced euqation are $(2, 3)$.

So the roots of the original equation are obtained by dividing the smaller root by the square root of 4, i.e., $(\frac{2}{2}, 3) = (1, 3)$.

Example

dvātriṁśat guṇito vargaḥ kaḥ sa-eko mūlado vada

Tell the number, the square of which when multiplied by 32 and increased by 1 gives a perfect square. The equation is

$32x^2 + 1 = y^2$.
Dividing the prakṛti by 16, we get $2x^2 + 1 = y^2$.

This has a solution $(2, 3)$.
Hence the solution of the original equation is $(\frac{2}{4}, 3)$.
Using bhāvanā, we get
$(2 * \frac{2}{4} * 3, 32 * \frac{2}{4}\frac{2}{4} * 3 * 3) = (3, 17)$.

Example

iṣṭa bhakto dvidhā kṣepa iṣṭonāḍhyo dalīkṛtāḥ
guṇa mūla hṛtaḥ cādyo hṛasva jyeṣṭe kramālpade

If the prakṛti(guṇa) is a square number, divide the constant argument by any optional number and write the quotient at 2 places. From one subtract the optional number and add it to the second. Divide both by 2. Dividing the first one by the square root of the guṇa gives the smaller root x and the second one gives the larger root.
For example, consider $16x^2 + 10 = y^2$. The guṇa is a perfect square.
Divide the constant say by 2. We get $\frac{10}{2} = 5$. Subtracting and adding 2 from 5 and then dividing it by 2, we get $\frac{5-2}{2}$ and $\frac{5+2}{2}$. Dividing the first one by square root of 16, we get $\frac{3}{2*4}$ as value of x, and $y = \frac{7}{2}$.

General Observations

While, both the methods provide a way to get a solution, it is not clear whether we will get the solution at all, and if yes, how many such interations are needed. Or in other words, how fast the solution will converge?

Bhāskarāchārya uses a combination of kuṭṭaka and bhāvanā for fast convergence.