A mathematical analysis of Pāṇini's *Śivasūtras*

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Abstract

In Pāṇini's grammar one finds the *Śivasūtras*, a table which defines the natural classes of phonological segments in Sanskrit by intervals. We present a formal argument which shows that, using his representation method, Pāṇini's way of ordering the phonological segments to represent the natural classes is optimal. The argument is based on a strictly set-theoretical point of view depending only on the set of natural classes and does not explicitly take into account the phonological features of the segments, which are, however, implicitly given in the way a language clusters its phonological inventory. Moreover, the argument is so general that it allows one to decide for each set of sets whether it can be represented with Pāṇini's method. Actually, Pāṇini had to modify the set of natural classes in order to define it by the *Śivasūtras* (the segment *h* plays a special role). We show that this modification was necessary and, in fact, the best possible modification. We discuss how every set of classes can be modified in such a way that it can be defined in a *Śivasūtra*-style representation.[1]

1 Pāṇini's *Śivasūtras*

Among grammars, Pāṇini's description of Sanskrit takes up an outstanding position. On the one hand, it is one of the oldest recorded grammars but also one of the most complete grammars of any language. The text of the grammar is rather accurately preserved, since it consists of *sūtras* designed for ritual repetition. On the other hand, this grammar attracts attention because it anticipates many structural and methodological approaches of modern linguistic theories.

> Modern linguistics acknowledges it as the most complete generative grammar of any language yet written, and continues to adopt technical ideas from it. [Kiparsky, 1994]

Or, as Bloomfield states about this grammar:

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[1] This approach fits naturally in the framework of Formal Concept Analysis, since the investigated graphs are formal concept lattices.

1.	a	i	u			Ṇ
2.				ṛ	ḷ	K
3.		e	o			Ṅ
4.		ai	au			C
5.	h	y	v	r		Ṭ
6.					l	Ṇ
7.	ñ	m	ṅ	ṇ	n	M
8.	jh	bh				Ñ
9.			gh	ḍh	dh	Ṣ
10.	j	b	g	ḍ	d	Ś
11.	kh	ph	ch	ṭh	th	
			c	ṭ	t	V
12.	k	p				Y
13.		ś	ṣ	s		R
14.	h					L

Table 1: Pāṇini's *Śivasūtras*

> The descriptive grammar of Sanskrit, which Pāṇini brought to its perfection, is one of the greatest monuments of human intelligence and an indispensable model for the description of languages.
> [Bloomfield, 1929]

The grammar gives a theoretical analysis of the spoken Sanskrit (*bhāṣā*) of the time of its origin (according to Kiparsky [1994] it dates from around 350 BC). As a whole, the grammar consists of four parts, of which the *Aṣṭādhyāyī* plays the central role, as it contains nearly 4000 rules governing how the elements defined in the other components can be used. The *Śivasūtras* form the component in which the phonological segments of the language and their grouping in natural phonological classes, designated by *pratyāhāras*, is defined (a short survey of the structure of Pāṇini's grammar can be found in Kiparsky [1994]). In the *Aṣṭādhyāyī* Pāṇini refers to the phonological classes in hundreds of rules.

The *Śivasūtras* define 42 phonological segments and consist of 14 *sūtras* (rows in table 1), each of which consists of a sequence of phonological segments (transcribed with small letters) bounded by a marker (transcribed with a capital letter), called *anubandha*. Phonological classes are denoted by abbreviations, called *pratyāhāras*, consisting of a phonological segment and an *anubandha*. The elements of such a class are defined by the *Śivasūtras* given in table 1 and are the continuous sequence of phonological segments starting with the given segment and ending with the last segment before the *anubandha*. Table 2 gives an example of a *pratyāhāra*.

A typical phonological rule found in the *Aṣṭādhyāyī* is **iko yaṇ aci**, the functional analysis of which is $[\text{ik}]_{\text{GEN}}[\text{yaṇ}]_{\text{NOM}}[\text{aci}]_{\text{LOC}}$. It is interpreted as iK $\rightarrow$ [yṆ]/_[aC]. This rule states that the vowels of the class iK= {i,u,ṛ,ḷ} are replaced by their nonsyllabic counterparts yṆ= {y, v, r, l} before a vowel aC= {a,i,u,ṛ,ḷ,e,o,ai,au}.

1.	a	**i**	**u**			Ṇ
2.				**ṛ**	**ḷ**	K
3.		**e**	**o**			Ṅ
4.		**ai**	**au**			C
5.	h	y	v	r		Ṭ

Table 2: Example of a *pratyāhāra*: iC = {i,u,ṛ,ḷ,e,o,ai,au}

Altogether, 285 *pratyāhāras* can be constructed, which is more than the 42 actually referred to by rules of the *Aṣṭādhyāyī*, but it is still a small number compared to the number of all classes that can be formed from the phonological segments, which is $2^{42} > 4 \cdot 10^{12}$.

As Kornai [1993] points out clearly, the task of characterizing a phonological system of a language is to specify the segmental inventory, phonological rules, and the set of natural classes of phonological segments to enable generalized rules.

> What is required is a clever notation that lets us characterize any such $R \subset S$, traditionally called a *natural class*, in a compact manner so that rules in terms of natural classes are just as easy, or perhaps even easier, to deal with as rules stated in terms of segments.
> [Kornai, 1993]

The set of natural classes is externally given by the phonological patterning of a language and it always meets two conditions: first, it is small compared to the set of unnatural classes and second, the set of natural classes is basically closed under intersection. These conditions have to be reflected in the notational device for the classes, too.

Kornai stresses that the representation device used for the notation of the classes must make it easier to use natural classes than unnatural ones (e.g. the complement of a natural class is generally unnatural). Contemporary phonological theories build up a structured system of phonological features which are used to characterize the natural classes. Instead of referring to phonological features in order to define a phonological class, Pāṇini refers to intervals in a linear order of the phonological segments. His method of defining the natural classes by *pratyāhāras*, denoting intervals of the *Śivasūtras*, meets the required conditions.

The phonological classes of a grammar are mutually related: classes can be subclasses of other classes, two or more classes can have common elements, etc. These connections are naturally represented in a hierarchy. A *Śivasūtra*-style representation encodes such connections in a linear form.[2] The linear representation method is so ingenious that it allows one to answer questions about the hierarchical relations of the classes without referring to the elements

[2]Since Pāṇini's grammar was designed for oral tradition, the linear form of the *Śivasūtras* was a prerequisite.

of the classes. For example, two classes have common elements if at least one interval boundary of the first lies in the interval of the second. An aim of this paper is to determine the conditions under which a set of sets in fact has a *Śivasūtra*-style linear representation.

The rest of the paper is organized as follows: In section 2 a general formalization of Pāṇini's *Śivasūtra*-style representation of phonological classes is given. Furthermore, the main s which will be answered in the course of the paper are raised. Section 3 explains how the Hasse-diagrams of sets of subsets determine whether a *Śivasūtra*-style representation of natural classes exists. Since some results of graph theory are needed, a brief introduction to planar graphs is given. Finally, in section 4 a procedure is presented which constructs a good *Śivasūtra*-style representation of a set of natural classes if it exists. This section ends with the proof that Pāṇini has chosen an optimal *Śivasūtra*-style representation. Furthermore, examples of feature-based analysis of the phonological system of German are stated and translated into a *Śivasūtra*-style representation. The whole approach is based only on a set- and order-theoretical investigation of the set of natural classes used in Pāṇini's grammar of Sanskrit. No external – phonological or methodological – arguments are involved.

2 General definitions and the main questions

Definition 2.1 *A* well-formed *Śivasūtra*-alphabet *(short* S-alphabet*) is a triple* $(\mathcal{A}, \Sigma, <)$ *consisting of a finite alphabet* $\mathcal{A}$ *and a finite set of markers* Σ *(such that* $\mathcal{A} \cap \Sigma = \emptyset$*), and a total order* $<$ *on* $\mathcal{A} \cup \Sigma$.

Definition 2.2 *A subset* T *of the alphabet* $\mathcal{A}$ *is* S-encodable in an S-alphabet $(\mathcal{A}, \Sigma, <)$ *iff there exists* $a \in \mathcal{A}$ *and* $M \in \Sigma$*, such that* $T = \{b \in \mathcal{A} | a \leq b < M\}$. aM *is called the* pratyāhāra *or* S-encoding *of* T *in* $(\mathcal{A}, \Sigma, <)$.

If two subsets $T_1, T_2 \in \mathcal{A}$ are S-encodable in the S-alphabet $(\mathcal{A}, \Sigma, <)$, then the intersection of the two sets $T_1 \cap T_2$ is S-encodable in $(\mathcal{A}, \Sigma, <)$, too. This is the case since, if the S-encoding of T_1 is $a_1 M_1$ and the S-encoding of T_2 is $a_2 M_2$, then the S-encoding of $T_1 \cap T_2$ is $\max(a_1, a_2) \min(M_1, M_2)$. This shows that the classes which are definable by S-alphabets meet the second condition of natural classes, namely that the intersection of two natural classes is a natural class itself. The first condition (i.e., that the set of natural classes is small compared to the set of unnatural classes is met, too, since the number of *pratyāhāras* is at most $\binom{n}{2} + n = \frac{n^2+n}{2}$ if the alphabet $\mathcal{A}$ consists of n elements. This number is for $n > 4$ smaller than half of the number of all possible classes, which is 2^n. Hence, the *pratyāhāras* form a suitable denotational device for defining natural classes.

Definition 2.3 *An S-alphabet* $(\mathcal{A}', \Sigma, <)$ corresponds *to a system of sets* $(\mathcal{A}, \Phi)$ *(where* Φ *is a set of subsets of* $\mathcal{A}$*) iff* $\mathcal{A} = \mathcal{A}'$ *and each element of* Φ *is S-encodable in* $(\mathcal{A}', \Sigma, <)$. *An S-alphabet which corresponds to* $(\mathcal{A}, \Phi)$ *is called an* S-alphabet

of $(\mathcal{A}, \Phi)$. *A system of sets for which a corresponding S-alphabet exists is said to be* S-encodable.

For example, take the set of subsets

$$\Phi = \{\{d, e\}, \{b, c, d, f, g, h, i\}, \{a, b\}, \{f, i\}, \{c, d, e, f, g, h, i\}, \{g, h\}\} \quad (1)$$

of the alphabet $\mathcal{A} = \{a, b, c, d, e, f, g, h, i\}$: it is S-encodable and

$$a\, b\, M_1\, c\, g\, h\, M_2\, f\, i\, M_3\, d\, M_4\, e\, M_5 \quad (2)$$

is one of the corresponding S-alphabets. The *pratyāhāras* of Φ are: dM_5, bM_4, aM_1, fM_3, cM_5 and gM_2.

Definition 2.4 *An S-alphabet* $(\mathcal{A}, \Sigma, <)$ *of* $(\mathcal{A}, \Phi)$ *is said to be* good *iff there exists no other S-alphabet* $(\mathcal{A}, \Sigma', <')$ *of* $(\mathcal{A}, \Phi)$ *such that the set of markers* Σ' *has fewer elements than* Σ.

Looking at Pāṇini's *Śivasūtras* it is striking that the phonological segment h occurs twice, namely in *sūtra* 5 and *sūtra* 14. To model this phenomenon we will introduce the concept of *enlarging* an alphabet by duplicating some of its elements.[3]

$\hat{\mathcal{A}}$ is said to be an *enlarged alphabet* of $\mathcal{A}$ if there exists a surjective map $\vartheta : \hat{\mathcal{A}} \to \mathcal{A}$. We can extend the map ϑ naturally to sets $\vartheta : \mathbf{Pot}(\hat{\mathcal{A}}) \to \mathbf{Pot}(\mathcal{A})$ with $\vartheta(\hat{\varphi}) = \{\vartheta(a) : a \in \hat{\varphi}\}$. It is clear that for every system of sets $(\mathcal{A}, \Phi)$ we can find an enlarged alphabet $\hat{\mathcal{A}}$ and a set of subsets $\hat{\Phi}$ of $\hat{\mathcal{A}}$ with $\Phi = \{\vartheta(\hat{\varphi}) : \hat{\varphi} \in \hat{\Phi}\}$ such that $(\hat{\mathcal{A}}, \hat{\Phi})$ is S-encodable.[4] To achieve such an S-encodable system of sets $(\hat{\mathcal{A}}, \hat{\Phi})$ we enlarge $\mathcal{A}$ and choose $\hat{\Phi}$ so that the elements of $\hat{\Phi}$ are disjoint. Then we arrange the sets of $\hat{\Phi}$ in a sequence and separate them by markers. The induced S-alphabet $(\hat{\mathcal{A}}, \hat{\Sigma}, \hat{<})$ obviously corresponds to $(\hat{\mathcal{A}}, \hat{\Phi})$. The following example illustrates this procedure: Let

$$\Phi = \{\{a, b\}, \{a, c\}, \{b, c\}\} \quad \text{and} \quad \mathcal{A} = \{a, b, c\} \quad (3)$$

be the system of sets for which an enlarged S-alphabet is searched. $(\mathcal{A}, \Phi)$ is not S-encodable since the elements of $\mathcal{A}$ cannot be linearly ordered in such a way that the elements of each member of Φ form an interval. We cam enlarge $\mathcal{A}$ by duplicating each of its elements ($\hat{\mathcal{A}} = \{a, a', b, b', c, c'\}$) and then we can take $\hat{\Phi} = \{\{a, b\}, \{a', c\}, \{b', c'\}\}$. Now

$$a\, b\, M_1\, a'\, c\, M_2\, b'\, c'\, M_3 \quad (4)$$

is an S-alphabet of $(\hat{\mathcal{A}}, \hat{\Phi})$ and therefore an enlarged S-alphabet of $(\mathcal{A}, \Phi)$.[5]

Since we always find a finite, enlarged S-alphabet of $(\mathcal{A}, \Phi)$, a minimally enlarged S-alphabet exists.

[3] Duplicating an element a means adding a new element a' to $\mathcal{A}$ and changing some of the occurrences of a in Φ to a'.

[4] An S-alphabet of $(\hat{\mathcal{A}}, \hat{\Phi})$ will sometimes be called an *enlarged S-alphabet of* $(\mathcal{A}, \Phi)$.

[5] This example only illustrates a general method of constructing an enlarged S-alphabet. It is clear that there are better enlargements, leading to more compact S-alphabets (e.g. $a\, b\, M_1\, c\, M_2\, a'\, M_3$).

Definition 2.5 *An enlarged S-alphabet* $(\hat{\mathcal{A}}, \hat{\Sigma}, \hat{<})$ *of* $(\mathcal{A}, \Phi)$ *is said to be* optimal *iff it fulfills the following conditions: First, there exists no other enlarged S-alphabet* $(\tilde{\mathcal{A}}, \tilde{\Sigma}, \tilde{<})$ *of* $(\mathcal{A}, \Phi)$*, the alphabet* $\tilde{\mathcal{A}}$ *of which has fewer elements than* $\hat{\mathcal{A}}$ *and furthermore, as a secondary condition,* $(\hat{\mathcal{A}}, \hat{\Sigma}, \hat{<})$ *is good.*

After this introduction of basic concepts the following questions will be investigated in the present paper:

1. Do Pāṇini's *Śivasūtras* form an optimal S-alphabet?
   Here we have to answer the questions:
   (a) Is the duplication of a phonological segment necessary?
   (b) If a duplicated element is necessary, does the duplication of the 'h' lead to an S-alphabet with as few markers as possible?
2. Is there a general method to decide whether a set of sets is S-encodable?
3. If a system of sets is S-encodable, is there a good method to construct a good corresponding S-alphabet?

Kiparsky [1991] discusses question 1 and shows that its affirmation follows from the principle of economy and the logic of the special case and the general case used in the construction of Pāṇini's whole grammar. I will answer all three questions, using a different approach. Furthermore, I will give some hints about how to find an enlargement of a set of sets leading to an optimal S-alphabet.[6]

3 The existence of *Śivasūtra*-style representations of systems of sets

3.1 A Brief introduction to the theory of planar graphs

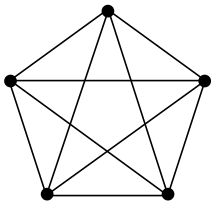
Throughout this paper we will need some basic knowledge about planar graphs, which will be briefly introduced in this section.[7] A *graph* G is a pair (V, E) consisting of a set of *vertices* V and a set of *edges* $E \subseteq V \times V$. *Paths* in graphs are defined in the natural way and *circles* are closed paths, as usual. *Directed graphs* are graphs the edges of which are directed.

A graph is a *plane graph* if its vertices are points in the Euclidean plane $\mathbf{R} \times \mathbf{R}$ and its edges are polygonal arcs in $\mathbf{R} \times \mathbf{R}$, such that neither a vertex nor a point of an edge lies in the inner part of another edge. The Euclidean plane $\mathbf{R} \times \mathbf{R}$ is subdivided by a plane graph into *faces* (areas). Exactly one of this faces, the *infinite face*, is of unlimited size.

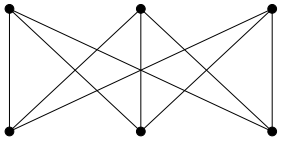
If a graph is isomorphic to a plane graph, it is said to be *planar*. One of the most important criteria for the planarity of graphs is the criterion of Kuratowski, which is based on the notion of minors of a graph. A graph M is said to be a

[6] Note that we showed that for every system of sets an optimal S-alphabet exists.

[7] The full details can be found in every introductory book on graph theory. In writing this paper Diestel [1997] proved to be especially helpful.



(a) K^5: complete graph with 5 vertices



(b) $K_{3,3}$: complete bipartite graph with $2 \cdot 3$ vertices

Figure 1: The graphs K^5 and $K_{3,3}$

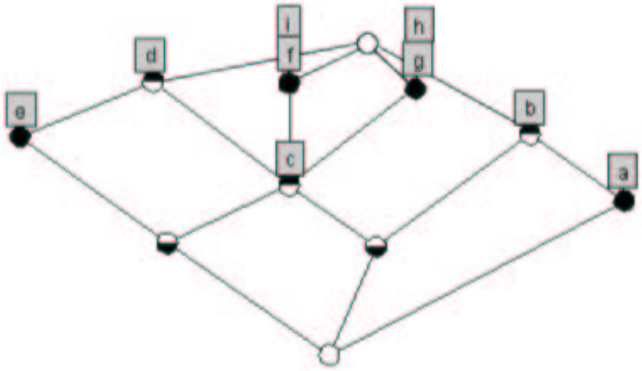


Figure 2: Hasse diagram of $(\mathcal{H}(\Phi), \supseteq)$, $\Phi = \{\{d, e\},\ \{b, c, d, f, g, h, i\},\ \{a, b\},\ \{f, i\},\ \{c, d, e, f, g, h, i\},\ \{g, h\}\}$

minor of a graph G if it can be arrived from G by first removing a number of vertices and edges from G and then contracting some of the remaining edges.

Proposition 3.1 (Criterion of Kuratowski) *A graph G is planar iff G contains neither a K^5 nor a $K_{3,3}$ as a minor (see figure 1).*

3.2 Plane Hasse-diagrams and S-encodability

Let $(\mathcal{A}, \Phi)$ be a system of sets as above, and let $\mathcal{H}(\Phi)$ be the set of all intersections of elements of $\Phi \cup \{\mathcal{A}\}$. $\mathcal{H}(\Phi)$ is partially ordered by the superset relation. A Hasse-diagram of a partially ordered set is a drawing of a directed graph whose vertices are the elements of the set and whose edges correspond to the upper neighbor relations determined by the partial order. The drawing must meet the following condition: if an element of the partially ordered set is an upper neighbor of another element, then its vertex lies above the vertex of the other one. In this paper we will stipulate that all edges are directed upwards.

Figure 2 shows a drawing of the Hasse-diagram $(\mathcal{H}(\Phi), \supseteq)$ of our example system of sets $(\mathcal{A}, \Phi)$ from (1). All Hasse-diagrams in this paper are labeled

economically as follows: for every $b \in \mathcal{A}$ the smallest set of $\mathcal{H}(\Phi)$ containing b (its b-set) is labeled. The set corresponding to a vertex of the diagram can be reconstructed by collecting all elements which are labeled to the vertex itself or to a vertex above. For example, in figure 2 the vertex labeled with c corresponds to the set $\{c, d, f, g, h, i\}$.

The Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ gives a first hint of the question whether $(\mathcal{A}, \Phi)$ is S-encodable:

Proposition 3.2 *If $(\mathcal{A}, \Phi)$ is S-encodable, then the Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ is a planar graph.*[8]

Let $(\mathcal{A}, \Sigma, <)$ be an S-encoding of $(\Phi, \mathcal{A})$. The proof of the proposition is based on a stepwise construction of a plane Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$, starting with the smallest sets. The smallest element of each set of $\mathcal{H}(\Phi)$ with respect to $(\mathcal{A}, \Sigma, <)$ determines the x-coordinate of this set in the Hasse-diagram. Its y-coordinate is given by the length of a maximal chain between the set and the empty set. It is possible to prove that the edges of the Hasse-diagram can be drawn without intersections.

As a corollary of proposition 3.2 it follows that a system of sets is not S-encodable whenever the Hasse-diagram of the corresponding set of intersections is not planar.

Together with Kuratowski's criterion 3.1 this answers question 1a, since figure 3 shows a section of the Hasse-diagram of the natural classes and their intersections, which has K^5 as a minor.[9] Hence, Pāṇini was forced to duplicate at least one of the phonological segments. But it remains to prove that h is the best candidate for the duplication; this discussion will be postponed.

The condition for S-encodable systems of sets given in proposition 3.2 is necessary but not sufficient, however. Figure 5 shows an example of a system of sets which is not S-encodable, although the Hasse-diagram of its intersection sets is planar. We need a second proposition to fully identify those systems of sets which are S-encodable.

If $(\mathcal{A}, \Phi)$ is a system of sets which is S-encodable, then the boundary of the infinite face of a plane Hasse-diagram of $(\mathcal{H}(\Phi) \setminus \emptyset, \supseteq)$ is called the *S-graph of* $(\mathcal{A}, \Phi)$. It can be shown that the S-graph of $(\mathcal{A}, \Phi)$ is fixed up to isomorphism, and is therefore independent of the chosen embedding of the Hasse-diagram in $\mathbf{R}^2$.

Proposition 3.3 *If $(\mathcal{A}, \Phi)$ is S-encodable, then the Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ is a planar graph and the S-graph of $(\mathcal{A}, \Phi)$ meets the following condition: For every $b \in \mathcal{A}$ the smallest set of $\mathcal{H}(\Phi)$ containing b (its b-set) is a vertex of the S-graph.*

[8] Due to the limited space, proofs are only sketched in this paper, but the results are illustrated by a number of examples.

[9] The emphasized lines in the figure mark a way to arrive at the minor K^5: remove all edges which are not emphasized and contract those edges which are marked by arrows. The sequence of graphs in figure 4 shows the result of each step of contraction.

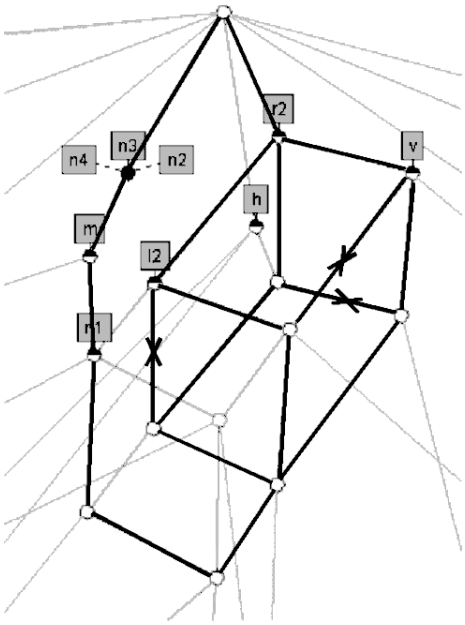


Figure 3: A section of the Hasse-diagram of the *pratyāhāras* used in the *Aṣṭādhyāyī* which has K^5 as a minor. The figure shows that the class memberships of the phonological segments h, v and l (denoted by *l2*) are independent of each other.

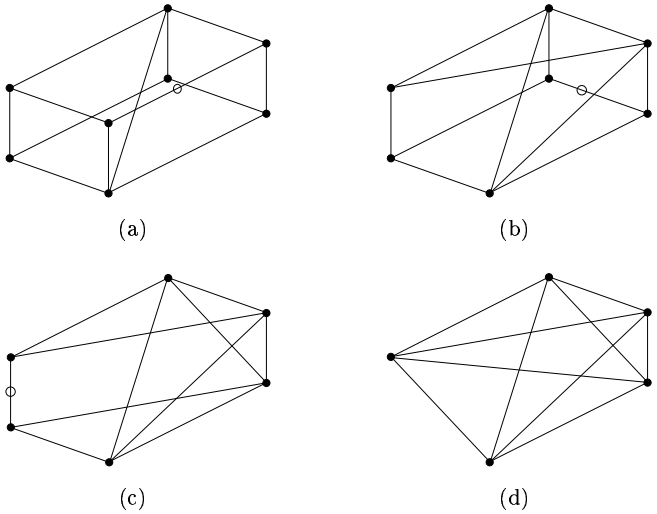


Figure 4: This sequence of graphs shows that the emphasized graph of figure 3 has K^5 as a minor. The small circles indicate the edges which will be contracted in the next step.

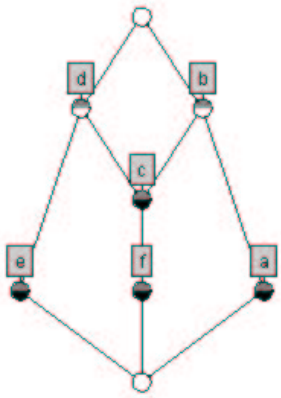


Figure 5: Hasse-diagram of the intersections of the sets $\{\{d,e\},\{a,b\},\{b,c,d\},\{b,c,d,f\},\{a,b,c,d,e,f\}\}$. It is plane, but there exists no corresponding S-alphabet.

The key idea of the proof of this proposition is to realize that, if for each $a \in \mathcal{A}$ the singleton $\{a\}$ is added to $\mathcal{H}(\Phi)$, the Hasse-diagram does not change much. Each new set $\{a\}$ creates a new vertex and an edge between $\{a\}$ and the a-set. If $(\mathcal{A},\Phi)$ is S-encodable, then the new Hasse-diagram is planar. And if the new Hasse-diagram is planar, the S-Graph of $(\mathcal{A},\Phi)$ contains all a-sets as vertices. The following section shows that if the conditions of proposition 3.3 are fulfilled, an S-alphabet of $(\mathcal{A},\Phi)$ exists.

In the example of figure 5 the f-set violates the condition of proposition 3.3 since in a plane Hasse-diagram of $(\mathcal{H}(\Phi)\setminus\emptyset,\supseteq)$ it would only touch inner faces. It is obvious that there exists no alternative embedding of this Hasse-diagram into $\mathbf{R}^2$ which would fulfill both conditions.

4 The construction of *Śivasūtra*-style representations of systems of sets

4.1 The boundary graph determines the S-alphabet

If $(\mathcal{A},\Phi)$ is a system of sets which is S-encodable, then an S-alphabet $(\mathcal{A},\Sigma,<)$ of $(\mathcal{A},\Phi)$ can be found as follows: Take the labeled S-graph of $(\mathcal{A},\Phi)$ and a path in it, that starts and ends at the vertex corresponding to $\mathcal{A}$. The path must meet the following conditions: First, for every $a \in \mathcal{A}$ the path passes the a-set at least once; second, none of the edges occurring more than once in the path is part of a circle in the S-graph. By looking at the S-graph as a subgraph of the directed Hasse-diagram, the edges of the path can be directed.

The S-alphabet, seen as a sequence of markers and elements of $\mathcal{A}$, can be constructed from the empty sequence by traversing the path from the beginning to the end: If a vertex is reached which is labeled with an a-set, then add a to the sequence, together with all other labels of the same vertex. If an edge

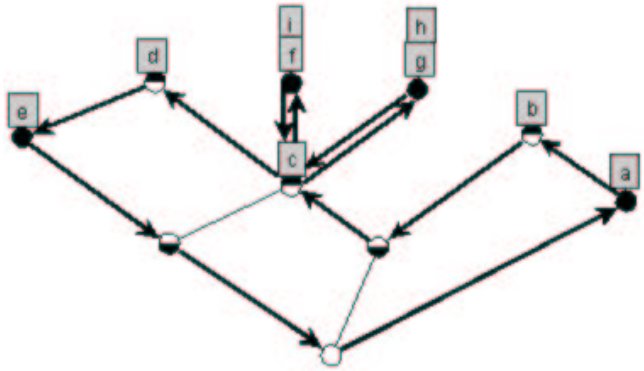


Figure 6: Hasse-diagram of example (1) with a possible path in its S-graph from which the S-alphabet $a\,b\,M_1\,c\,g\,h\,M_2\,f\,i\,M_3\,d\,M_4\,e\,M_5$ can be achieved.

is passed whose direction contradicts the traversal direction, a new, previously unused, marker element is added to the sequence, unless the last added element is already a marker. Finally, after the end of the path is reached, revise the sequence as follows: If an element of $\mathcal{A}$ appears more than once in the sequence, delete all occurrences of it except one, and if two markers happen to occur next to each other, remove one.

Applied to our small example (1) and the plane graph of its Hasse-diagram given in figure 2, we may choose the path illustrated in figure 6, which fulfills the required conditions. Traversing the path, we pass first the a-set and the b-set without using an edge against its destined direction. Now we move downwards and violate the direction of the edge, and therefore we have to add a marker to our sequence, so that it starts with $a\,b\,M_1$. Now moving upwards we collect the c and the g, but since the g- and the h-sets are identical we also have to collect the h. After this we move downwards again, and that is why we add a new marker. We again reach the c-set and add c a second time to our sequence. So far our sequence is $a\,b\,M_1\,c\,g\,h\,M_2\,c$, and if we continue we end up with the S-alphabet depicted in (2).

Note that this procedure does not yield a unique S-alphabet since we have several decisions to make: (a) If a vertex is labeled with more than one element, their order in the S-alphabet is arbitrary; (b) the ultimately deleted elements are arbitrarily chosen; (c) from the vertex labeled c we can either go to the vertex labeled gh or if; (d) the path can be traversed clockwise or anti-clockwise.

It can be shown that every good S-alphabet of $(\mathcal{A}, \Phi)$ can be constructed in this way by finding a path which fulfills the conditions and violates the direction of the edges as seldom as possible. This proves our main theorem and answers questions 2 and 3.

Theorem 4.1 *$(\mathcal{A}, \Phi)$ is S-encodable iff the Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ is isomorphic to a plane graph G and for every $b \in \mathcal{A}$ the b-set lies at the boundary of the infinite face of G', where G' is obtained from G by removing the vertex*

of the empty set and all corresponding edges.

If $(\mathcal{A}, \Phi)$ *is S-encodable, then all good S-alphabets* $(\mathcal{A}, \Sigma, <)$ *of* $(\mathcal{A}, \Phi)$ *can be constructed systematically.*

4.2 Pāṇini's *Śivasūtras* are optimal

Figure 7 shows the Hasse-diagram, with duplicated h, corresponding to the *pratyāhāras* which Pāṇini uses in his *Aṣṭādhyāyī*; the duplication of h is denoted by h_{-}.[10] The 42 *pratyāhāras* actually used by Pāṇini in the *Aṣṭādhyāyī* are marked in the figure with white boxes. The black and the striped rectangles next to some of the vertices mark the places where markers have to be added, depending on the traversal direction (black: anti-clockwise [14 markers], striped: clockwise [17 markers]). It is obvious that no S-encoding can have less than 14 markers and the good S-alphabets are the various combinatorial variants of

$$\langle \text{a,i,u}, M_1, \text{ṛ,ḷ}, M_2, \{\langle\{\text{e,o}\}, M_3\rangle, \langle\{\text{ai,au}\}, M_4\rangle\}, \text{h,y,v,r}, M_5, \text{l}, M_6,$$
$$\text{ñ,m,}\{\text{ṅ,ṇ,n,}\}, M_7, \text{jh,bh}, M_8, \{\text{gh,ḍh,dh}\}, M_9, \text{j,}\{\text{b,g,ḍ,d}\}, M_{10},$$
$$\{\text{kh,ph}\}, \{\text{ch,ṭh,th}\}, \{\text{c,ṭ,t}\}, M_{11}, \{\text{k,p}\}, M_{12}, \{\text{ś,ṣ,s}\}, M_{13}, \text{h}, M_{14}\rangle\,.$$

Kiparsky [1991] argues in detail that the order chosen by Pāṇini out of the set of possibilities is unique if one requires a subsidiary principle of restrictiveness.

So far we have answered question 1a and questions 2 and 3 in the affirmative. Hence, we have argued that Pāṇini was forced to enlarge the alphabet, but it remains to show why duplicating the h is the best choice. If h is entirely removed from the natural classes, then the good S-alphabet has only one marker less, namely 13.

In the natural classes, the occurrence of h, v and l are independent of each other; since there exists a natural class – or an intersection of natural classes – for each subset of $\{h, l, v\}$ which contains the elements of the subset but no element of its complement in $\{h, l, v\}$. Therefore, the class memberships of the segments h, l and v are independent of each other. A Hasse-diagram which contains three independent elements has K^5 as a minor and is therefore not planar (see figure 3 and 4)). Triples of three independent elements are called K^5-triples. Hence, to get a planar Hasse-diagram it is necessary to duplicate at least one element of each triple.

Looking at the *pratyāhāras* used in the *Aṣṭādhyāyī* we find 249 K^5-triples; each of them contains h, and no other element is contained in each of them. Hence, to avoid the duplication of h it would be necessary to duplicate more than one element. For this reason there is no other choice then duplicating h in order to get an optimal S-alphabet corresponding to Pāṇini's *pratyāhāras*.

Summarizing, the all questions at issue have to be affirmed. Pāṇini's method of representing hierarchical information in a linear form is an interesting field of further investigations. Also the fact that one does not need to refer to phonological features explicitly in order to define phonological classes is remarkable.

[10] The drawing was done by the tool "Concept Explorer" written by Sergey Yevtushenko, which can be found at `http://www.sourceforge.net/projects/conexp`.

	p	b	f	v	p̆	m	t	d	s	z	c	n	l	r	š	ç	J	k	g	x	ɣ	ŋ	R
[nasal]						×						×										×	
[obstruent]	×	×	×	×	×		×	×	×	×	×				×	×	×	×	×	×	×		
[low]																							
[high]															×	×	×	×	×	×	×	×	
[back]																	×	×	×	×	×	×	
[anterior]	×	×	×	×	×	×	×	×	×	×	×	×	×	×									
[coronal]							×	×	×	×	×	×	×	×	×								
[continuant]			×	×				×	×			×	×	×	×	×			×	×		×	
[fricative]			×	×	×			×	×	×		×	×	×	×	×			×	×		×	
[strident]			×	×	×			×	×	×					×	×	×		×	×			
[voice]		×		×		×		×		×		×	×	×			×		×		×	×	×
[lateral]													×										

Table 3: Classification of the consonants of German, taken from Wurzel [1981].

	i	ɪ	y	ʏ	e	ɛ	ø	œ	æ	u	ʊ	ɯ	o	ɔ	a
[back]										×	×	×	×	×	×
[high]	×	×	×	×						×	×	×			
[low]									×						×
[tense]	×		×		×		×			×			×		
[round]			×	×			×	×		×	×		×	×	

Table 4: Classification of vowels, taken from Hall [2000]. An optimal corresponding enlarged S-alphabet is $\mathbf{\varepsilon ɪyʏuʊɯM_1 ɔaM_2 æM_3 ɔʊʏœøuyM_4 ieM_5}$. The *pratyāhāra* of the class of back vowels is $\mathbf{uM_2}$, the high vowels are denoted by $\mathbf{iM_1}$, the low vowels by $\mathbf{aM_3}$, the tense vowels by $\mathbf{oM_5}$ and the round vowels by $\mathbf{ɔM_4}$.

Kornai [1993] points out that Pāṇini's approach is generalized by *feature geometry*, and that it is genuinely weaker than the latter. Although Kornai argues that the power of feature geometry is needed in order to get the proper set of natural classes of a phonological system, for some special tasks like describing the set of *major class features* a *Śivasūtra*-style analysis seems to be quite appropriate.

However, after my investigations of several analyses of the phonological system of different languages, Pāṇini's phonological theory of Sanskrit seems to have a special, rare property: The 249 K^5-triples of phonological elements created by the *pratyāhāras* used in the *Aṣṭādhyāyī* turn out to be a typical number of such triples, but the fact that each of these triples contains the element h is extraordinary. Table 3 shows an analysis of the natural classes of the consonants of German taken from Wurzel [1981]. The twelve simple natural classes and their intersections yield 290 K^5-triples. But in any case, one needs to duplicate more than 8 of the 23 phonological elements in order to get a corresponding S-alphabet. A classification of vowels taken from Hall [2000] is given in table 4. Here, although the number of K^5-triples is only 126 one needs to duplicate 7 elements in order to eliminate all K^5-triples. A resulting S-alphabet is, e.g., $\varepsilon ɪyʏuʊɯM_1 ɔaM_2 æM_3 ɔʊʏœøuyM_4 ieM_5$. Up to now, the author only found

	a	i	u	ü	e	o	ö
[front]		×		×	×		×
[round]			×	×		×	×
[mid]					×	×	×
[low]	×						

Table 5: Example classification of vowels. An optimal corresponding enlarged S-alphabet is $\mathbf{uoM_1eM_2iM_3aM_3}$.

the simple analysis of German vowels given in table 5, which succeeds with a single duplication in an optimal S-alphabet ($u\ddot{u}o\ddot{o}M_1eM_2i\ddot{u}M_3aM_4$).

Finally, it should be emphasized again: The approach presented is so general that it is not limited to the domain of phonology.

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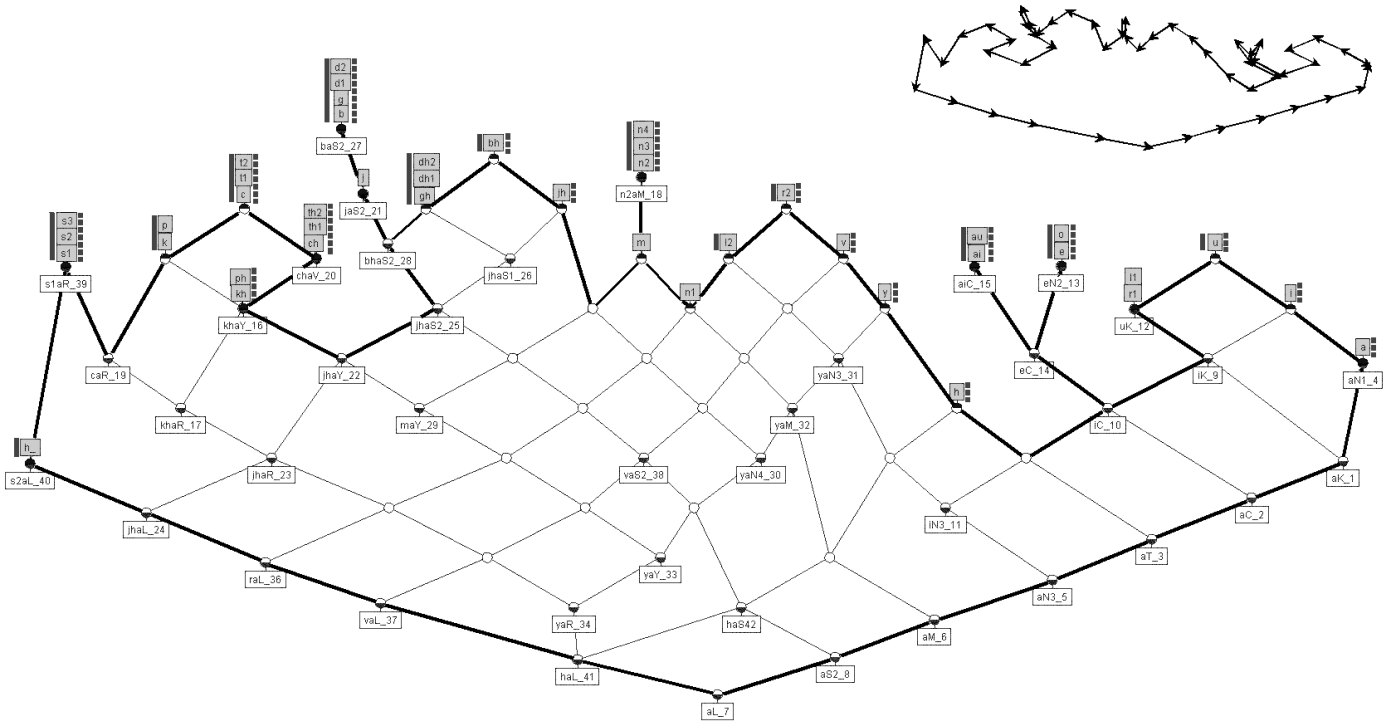


Figure 7: Hasse-diagram of the of the *pratyāhāras* used in the *Aṣṭādhyāyī*. The denotations in the figure are as follows: $h_$ is the duplicate of h and r1:ṛ, l1:ḷ, r2:r, l2:l, n1:ñ, n2:ṅ, n3:ṇ, n4:n, dh1:ḍh, dh2:dh, d1:ḍ, d2:d, th1:ṭh, th2:th, t1:ṭ, t2:t, s1:ś, s2:ṣ, s3:s.
The white boxes mark the 42 phonological classes which Pāṇini uses in the *Aṣṭādhyāyī*. The small figure on top shows the path in the S-graph which one has to choose in order to construct Pāṇini's *Śivasūtras*.