Mathematics and Astronomy in India before 300 BCE

Amartya Kumar Dutta[a] and M.S. Sriram[b]

[a]Stat-Math Unit, Indian Statistical Institute,
203, B.T. Road, Kolkata, 700 108, India. amartya.28@gmail.com
[b]Department of Theoretical Physics, University of Madras, Guindy Campus,
Chennai 600 025, India. sriram.physics@gmail.com

1 Introduction

Our knowledge about the mathematical accomplishments in India prior to 300 BCE is derived primarily from ancient Sanskrit texts, especially the Vedic and the *Vedāṅga* treatises. Though the Vedic *Saṁhitā*s and *Brāhmaṇa*s are not technical texts, their number-vocabulary and references to astronomical phenomena give us glimpses of early Vedic insights into mathematics and astronomy.

Some of the technical knowledge of the Vedic era is preserved in the six *Vedāṅga*s (literally, limbs of the Vedas): *śikṣā* (phonetics), *chandas* (prosody), *vyākaraṇa* (grammar), *nirukta* (etymology), *jyotiṣa* (astronomy) and *kalpa* (rituals). Among the mathematical knowledge recorded in the *Vedāṅga*s, special mention may be made of (i) the geometry results in the portion of the *kalpa* known as the *Śulba-sūtra*s, (ii) certain combinatorial algorithms in the *Chandaḥ-sūtra* of Piṅgalācārya, and (iii) a formalised calendrical system with a five-year *yuga* and intercalary months that has been described in the *Vedāṅga Jyotiṣa*.

That the study of mathematics was given an elevated status in India from at least the later Vedic period, can be seen from certain passages of *Upaniṣad*ic literature. In an episode in the *Chāndogya Upaniṣad* (7.1.2.4), Nārada mentions *nakṣatra-vidya* (astronomy) and *rāśi-vidya* (mathematics) among the disciplines of knowledge he had studied. Such branches of *aparāvidyā* (worldly knowledge) were considered helpful adjuncts to *parāvidyā* (spiritual knowledge). The importance of mathematics is emphasised in *Vedāṅga Jyotiṣa* (*Yajurved*ic recension 4):

> As are the crests on the head of a peacock, as are the gems on the hoods of a snake, so is *gaṇita* (mathematics) at the top of the *śāstra*s known as the *Vedāṅga*.

Jaina and Buddhist traditions too had a high esteem for the culture of mathematics. As in Vedic tradition, mathematics and astronomy were relevant for the Jaina ceremonies. One of the four branches of Jaina religious literature is *gaṇitānuyoga* (exposition of the principles of mathematics). A mastery over *saṁkhyāna* (arith-

metic) and *jyotiṣa* (astronomy) is stated to be one of the principal attainments of a Jaina priest. The *Sthānāṅga-sūtra* considers mathematics to be *sūkṣma* (subtle). The *Sūtrākṛtāṅga-sūtra* regards geometry as the "lotus of mathematics".[1] In Buddhist literature too, the science of numbers and calculations (*saṁkhyāna*, *gaṇana*), is ranked among the noblest of the arts.

In this article, we shall highlight some of the significant mathematical and astronomical concepts that occur in Indian treatises prior to 300 BCE. Our main emphasis will be on: (1) the number-vocabulary and arithmetical references in Vedic texts; (2) geometric constructions from the *Śulba-sūtra*s and their underlying algebraic principles; (3) enumeraive methods in *Chandaḥ-sūtra* and other texts; (4) astronomy in earlier Vedic literature, in the *Vedāṅga Jyotiṣa*, and a few subsequent works.

Indian mathematics and astronomy are algorithmic in nature.[2] The roots of this algorithmic approach can be found in the pre-300 BCE treatises discussed in our article.

2 Decimal system and Arithmetic in Vedic literature

In this section, we shall discuss the Vedic number vocabulary, its impact on the mathematical developments in subsequent centuries and give illustrations of arithmetical knowledge from Vedic literature.

Numbers are described verbally in a decimal (i.e., base ten) system in the *Ṛgveda* (the oldest Vedic treatise), in all other Vedic treatises, and in all subsequent Indian texts. The nomenclature in *Ṛgveda* involves *single-word* terms for the nine primary numbers: *eka* (1), *dvi* (2), *tri* (3), *catur* (4), *pañca* (5), *ṣaṭ* (6), *sapta* (7), *aṣṭa* (8) and *nava* (9); the first nine multiples of ten: *daśa* (10), *viṁśati* (20), *triṁśat* (30), *catvāriṁśat* (40), *pañcāśat* (50), *ṣaṣṭi* (60), *saptati* (70), *aśīti* (80) and *navati* (90), and the first four powers of ten: *daśa* (10), *śata* (10^2), *sahasra* (10^3) and *ayuta* (10^4). For compound numbers, the above names are combined as in our present verbal decimal terminology; e.g., "seven hundred and twenty" is expressed in *Ṛgveda* (1.164.11) as *sapta śatāni viṁśatiḥ*.

[1] For more details, see B. Datta (a), "The Jaina School of Mathematics", 116; 123–124.

[2] In fact, the word "algorithm" refers, etymologically, to the computational approach of ancient Indians. Thanks to the exposition of the Indian decimal system and computational arithmetic by Al-Khwārizmī (c. 820 CE), operations based on the decimal system acquired the name *algorismus*, derived from Algoritmi, the Latin form of Al-Khwārizmī. Thus emerged "algorism", which later became "algorithm"!

In the *Vājasaneyī Saṁhitā* (17.2) of *Śukla Yajurveda*, Medhātithi records single-word terms for progressively higher powers of ten: *eka* (1), *daśa* (10), *śata* (10^2), *sahasra* (10^3), *ayuta* (10^4), *niyuta* (10^5), *prayuta* (10^6), *arbuda* (10^7), *nyarbuda* (10^8), *samudra* (10^9), *madhya* (10^{10}), *anta* (10^{11}), *parārdha* (10^{12}). Medhātithi's terms occur with some variations, sometimes with further extensions, in other *Saṁhitā* and *Brāhmaṇa* texts. Terms for much higher powers of ten are mentioned in subsequent Jaina and Buddhist texts and in the epic *Rāmāyaṇa*. When convenient, centesimal (multiples of 100) scales have been used in India for expressing numbers larger than thousand — the *Ṛgveda* (1.53.9) describes 60099 as *ṣaṣṭiṁ sahasrā navatiṁ nava* (sixty thousand ninety nine). The *Taittirīya Upaniṣad* (2.8) adopts a centesimal scale to describe different orders of bliss and mentions *Brahmānanda* (the bliss of Brahman) to be 100^{10} times a unit of human bliss; there is a similar reference in the *Bṛhadāraṇyaka Upaniṣad* (4.3.33).

To appreciate the merit of the Sanskrit system of having a *single-word* term for *each* power of ten (up to some large power), note that even the present English terminology of using auxiliary bases like 'thousand' and 'million' is less satisfactory for expressing very large numbers in words. This is effectively illustrated by G. Ifrah by comparing the verbal representations of the number 523622198443682439 in English, Sanskrit and other systems.[3]

The transition from the oral Vedic decimal nomenclature to the written decimal place-value notation[4] would have required three ideas: (i) the replacement of the single-word terms for the nine primary numbers by single symbols called 'digits' or 'numerals' (*aṅka* in Sanskrit), (ii) the suppression of the single-word terms for powers of ten, and (iii) the use of a zero-symbol wherever needed to denote the absence of any of the above digits. Thus, the *single-word* terms for "powers of ten" play the role of, and are a verbal manifestation of, the abstract place-value principle.[5] In retrospect, a momentous step had been taken by ancient Vedic seers (or their unknown predecessors) when they imparted *single-word names* to successive powers of ten, thereby sowing the seeds of the decimal 'place-value' principle.

The decimal notation with zero had evolved in India during an unknown pe-

[3] G. Ifrah, *The Universal History of Numbers*, 428–429.

[4] The principle by which a digit d in the rth position (place) from the right is imparted the place-value $d \times 10^{r-1}$. For instance, in 1947, the symbol 1 acquires the place-value one thousand (1×10^3), 9 acquires the value nine hundered (9×10^2), etc. The Sanskrit term for "place" is *sthāna*.

[5] See B. Bavare and P.P. Divakaran, "Genesis and Early Evolution of Decimal Enumeration: Evidence from Number Names in Ṛgveda".

riod of its history, but within the early centuries of the Common Era. In the epic *Mahābhārata* (3.134.16), there is an incidental allusion to the decimal notation during the narration of a tale (3.132–134) involving ancient names like Uddālaka, Śvetaketu, Aṣṭāvakra, Janaka, et al, whose antiquity go back to the *Brāhmaṇa* phase of the Vedic era. Some Sanskrit scholars see in the term *lopa* (elision, disappearance, absence) of Pāṇini's grammar treatise *Aṣṭādhyāyī*, a concept analogous to zero as a marker for a non-occupied position. In the prosody-text *Chandaḥ-sūtra*, Piṅgalācārya (c. 300 BCE) gives instructions involving the use of *dvi* (two) and *śūnya* (zero) as distinct labels. The choice of the labels suggests the prevalence of the mathematical zero and possibly a zero-symbol by his time.

The decimal system (both in its verbal and notation forms) expresses any natural number as a polynomial-like sum $10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$, where $a_0, a_1, \ldots, a_n$ are numbers between 0 and 9. Such a representation involves recursive applications of the well-known 'division algorithm' that pervades the later Greek, Indian and modern mathematics. The mathematical sophistication of the system can be glimpsed from the fact that its discovery required a realisation of the above principles.

The decimal system is largely responsible for the excellence attained by Indian mathematicians in the fields of arithmetic, algebra and astronomy. The dormant algebraic character of the decimal system influenced the algebraic thinking of mathematicians in ancient India like Brahmagupta (who defined the algebra of polynomials in 628 CE) and Mādhavācārya (who investigated the power series expansions of trigonometric functions in 14th century CE). Its influence in the development of algebra in modern Europe can be seen from a statement of Isaac Newton in 1671 CE, emphasizing that the arithmetic of the decimal system provides a model for developing operations (addition, multiplication, root extraction, etc.) with algebraic expressions in variables (like polynomals and power series).[6] More recently, we see S.S. Abhyankar, a great algebraist of the 20th century, acknowledging the idea of decimal expansion in a technical innovation in his own research.[7]

Thanks to the decimal system, Indians developed efficient methods for the basic arithmetic operations which were slight variants of our present methods. The methods are described in post-Vedic treatises on mathematics but incidental references show that all the fundamental operations of arithmetic were performed during

[6] Newton's statement is quoted in P.P. Divakaran, "Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics", 296.

[7] See A.K. Dutta (d), "Was there Sophisticated Mathematics during Vedic Age?", 105.

the Vedic time. For instance, in a certain metaphysical context, it is mentioned in *Śatapatha Brāhmaṇa* (3.3.1.13) that when a thousand is divided into three equal parts, there is a remainder one.[8] A remarkable allegory in the *Śatapatha Brāhmaṇa* (10.24.2.2-17) lists all the factors of 720.

The *Pañcaviṁśa Brāhmaṇa* (18.3) describes a list of sacrificial gifts forming a geometrical progression (G.P.):

$$12, 24, 48, 96, 192, \ldots, 49152, 98304, 196608, 393216.$$

The *Śatapatha Brāhmaṇa* (10.5.4.7) mentions, correctly, the sum of an arithmetical progression (A.P.):

$$3(24 + 28 + 32 + \ldots \quad \text{to} \quad 7 \quad \text{terms}) = 756.$$

The *Bṛhaddevata*[9] gives the sum:

$$2 + 3 + 4 + ... + 1000 = 500499.$$

As mentioned in Section 4, the *Chandaḥ-sūtra* of Piṅgalācārya yields the formula for the sum of the G.P. series: $2 + \ldots + 2^n = 2^{n+1} - 2$. The following numerical example of this G.P. series can be seen in the Jaina treatise *Kalpasūtra* (c. 350 BCE) of Bhadrabāhu:

$$1 + 2 + 4 + ... + 8192 = 16383.$$

The incidental occurences of correct sums of such series in non-mathematical texts suggest that general formulae for series were known at least from the time of the *Brāhmaṇa*s.[10]

In the *Baudhāyana Śulba*, there are examples of operations with fractions like[11]

$$7\tfrac{1}{2} \div (\tfrac{1}{5})^2 = 187\tfrac{1}{2};\ 7\tfrac{1}{2} \div (\tfrac{1}{15} \text{ of } \tfrac{1}{2}) = 225;\ \sqrt{7\tfrac{1}{9}} = 2\tfrac{2}{3};\ (3 - \tfrac{1}{3})^2 + (\tfrac{1}{2} + \tfrac{10}{120})(1 - \tfrac{1}{3}) = 7\tfrac{1}{2}.$$

The structure of the Sanskrit number-vocabulary (culminating in a perfected decimal notation) enabled Indians to express large numbers effortlessly, right from the Vedic Age. This traditional facility with large numbers enabled Indians to work

[8] The problem is also alluded to in the earlier *Ṛgveda* (6.69.8) and the *Taittirīya Saṁhitā* (3.2.11.2).

[9] A treatise on Vedic deities ascribed to Śaunaka, a venerated Vedic seer. A.A. Macdonell places the text as being composed prior to 400 BCE.

[10] Explicit statements of the general formulae for the sum of A.P. and G.P. series occur later in the works of Āryabhaṭa (499 CE) and Śrīdharācārya (750 CE) respectively.

[11] For details, see B. Datta (c), *The Science of the Śulba*, Chapter XVI.

with large time-frames in astronomy (like cycles of 4320000 years) which helped them obtain strikingly accurate results.[12] Post-Vedic Indian algebraists could venture into problems of finding *integer* solutions of linear and quadratic equations which often involve large numbers.[13] The traditional preoccupation with progressively large numbers, that was facilitated by the decimal system, created an environment that was conducive for the introduction of the infinite in Indian mathematics.[14]

The Vedic number system is the first known example of recursive construction. Recursive principles dominate Indian mathematical thought and are prominent features of some of its greatest achievements like the solutions of indeterminate equations and the work of the Kerala school.[15]

3 Geometry and Geometric Algebra in *Śulba-sūtra*s

The *Śulba-sūtra*s of Baudhāyana, Mānava, Āpastamba and Kātyāyana give a compilation of principles in geometry that were used in designing the altars (called *vedi* or *citi*) where the Vedic sacrifices (*yajña*) were to be performed; the *Baudhāyana Śulba-sūtra* being the most ancient.[16] The *Śulba* texts are the outcome of a long and sophisticated tradition of geometry and brick technology. The major *Śulba* results and constructions are (known and) applied in the earlier *Śatapatha Brāhmaṇa* and the still earlier *Taittirīya Saṁhitā* of the *Yajurveda*.[17] It has also been surmised that the *Śulba* geometry and brick technology have links with the Harappan civilization.[18]

[12] For instance, Āryabhaṭa estimated that the Earth rotates around its axis in 23 hours 56 minutes and 4.1 seconds, which matches the modern estimate (23 hours 56 minutes 4.09 seconds).

[13] For instance, the smallest pair of integers satisfying $61x^2 + 1 = y^2$ is $x = 226153980$, $y = 1766319049$. And this example occurs in the Algebra treatise *Bījagaṇita* (1150 CE) of Bhāskarācārya. (See A.K. Dutta (b), Kuṭṭaka, Bhāvanā and Cakravāla, for further details.)

[14] Indian algebraists like Āryabhaṭa and Brahmagupta (628 CE) had a mastery over indeterminate equations with infinitely many solutions, Bhāskarācārya (1150 CE) introduced an algebraic concept of infinity and also worked with the infinitesimal in the spirit of calculus, and then there was the spectacular work on infinite series by Mādhavācārya.

[15] A.K. Dutta (b), op.cit.; P.P. Divakaran, op.cit.; K.V. Sarma, K. Ramasubramanian, M.D. Srinivas and M.S. Sriram, *Gaṇita-Yukti-Bhāṣā of Jyeṣṭhadeva*.

A detailed history of the decimal system in India is given in the source-book of B. Datta and A.N. Singh, and a brief overview is presented in the article by A.K. Dutta for the Springer Encyclopaedia. The system is also discussed in the article by B. Bavare and P.P. Divakaran, other articles by A.K. Dutta, and the book by Ifrah.

[16] The language of the *Śulba-sūtra*s is pre-Pāṇinian. Most historians of mathematics assign for Baudhāyana a date within 800–500 BCE. The archaic Brāhmaṇa style of Baudhāyana is suggestive of an early date like 800 BCE.

[17] B. Datta (c), op.cit., 25-40.

[18] G.G. Joseph, *Indian Mathematics*, 16. Also see M. Danino, *The Lost River*, 241-242 and F. Staal, Greek and Vedic Geometry, 119-121.

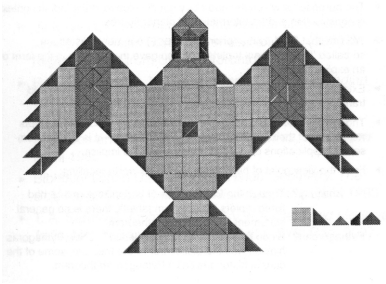


Figure 1: *śyenacit.*

The Vedic altars had rich symbolic significance and their designs were often intricate. For instance, the *śyenacit* in Figure 1 has the shape of a falcon in flight (a symbolic representation of the aspiration of the spiritual seeker soaring upward); the *kūrmacit* is shaped as a tortoise, with extended head and legs, the *rathacakracit* as a chariot-wheel with spokes, and so on. Further, the Vedic tradition demanded that these constructions are executed with perfection — the accuracy had to be meticulous — and the demand was met through remarkably sophisticated geometric innovations involving "exact constructions". Central to these constructions is Baudhāyana's version of the celebrated "Pythagoras Theorem", a result of unknown antiquity named after the mystic-mathematician Pythagoras (c. 540 BCE).

The familiar version of the "Pythagoras Theorem", recorded by Euclid in his *Elements* (c. 300 BCE), states that the square of the hypotenuse PR of a right-angled triangle PQR equals the sum of the squares of the other two sides PQ and QR. This result is a cornerstone of plane Euclidean geometry with applications throughout history. *Baudhāyana Śulba-sūtra* (1.48) states the following equivalent version: "The square on the diagonal AC of a rectangle ABCD is equal in area to the sum of the squares on the two adjacent sides AB and BC." This version is also stated by Āpastamba (1.4) and Kātyāyana (2.11).

The *Śulba-sūtra*s mention several rectangles the lengths of whose adjacent sides a, b and the length of each diagonal c are all integers (or rational numbers), i.e., (a, b, c) are integral (or rational) triples satisfying $a^2 + b^2 = c^2$. The triples (3,4,5), (5,12,13), (7,24,25), (8,15,17), (12,35,37) and some of their multiples (like (15, 36, 39)) occur in the *Śulba-sūtra*s.[19] The rectangle with sides 15 and 36 is highlighted not only in the *Śulba-sūtra*s but also in the much more ancient *Taittirīya Saṁhitā*

[19]See B. Datta (c), op.cit., 124, for more examples.

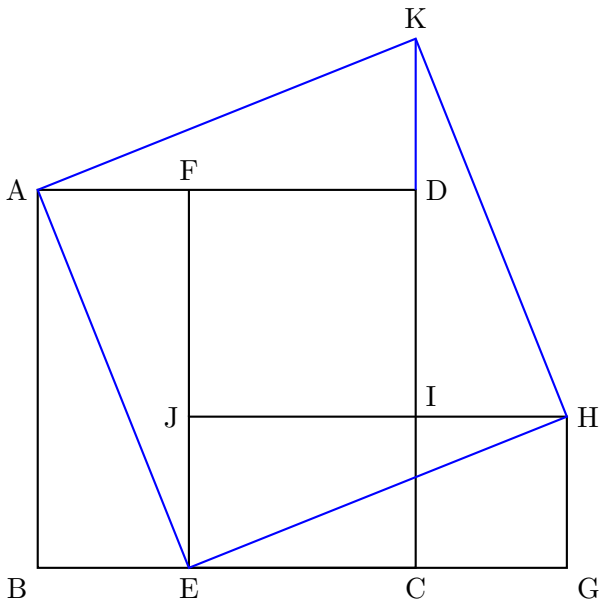


Figure 2: Baudhāyana's construction of a square equal in area to the sum of two squares.

(2.4.5) and the *Śatapatha Brāhmaṇa* (3.5.1.1; 10.2.3.4).

The theorem for the square on a diagonal would play a pivotal role in much of ancient Indian geometry and trigonometry. Some of the *Śulba* constructions indicate that the Vedic savants knew why the theorem holds. As an illustration, we show in Figure 2, Baudhāyana's construction of a square AEHK equal in area to the sum of the areas of two given squares ABCD and ICGH: E is chosen to be the point on BC for which BE = IH and the square AEHK is constructed on AE. That AEHK has the desired area is readily seen by observing that AEHK can be obtained by cutting off the triangles ABE and EGH from the original squares and placing them as triangles KIH and ADK in the new square.

This construction, described by Baudhāyana in *sūtra* 1.50 (shortly after the statement of his theorem for the square on the diagonal in *sūtra* 1.48) is itself a demonstration of the theorem! For, it shows that the square AEHK on the diagonal AE of the rectangle ABEF is equal in area to the sum of the square ABCD on the side AB and the square ICGH on the side CG (=BE).

A striking feature of the *Śulba* geometry is the abundance of "exact" constructions of the "straightedge-and-compass" type that makes our present high-school Euclidean geometry appear so formidable to a large section of students. These

constructions demand mathematical rigour and do not allow measurements.[20] For instance, to obtain a square whose area equals the sum of the areas of the square ABCD and the square ICGH, one could have measured the length a of AB, the length b of IC, then mark out a side of length $\sqrt{a^2+b^2}$ (which will often be an irrational number even when a and b are rational numbers) and draw a square on it. But all these steps would have involved approximations. The purely geometric construction from *Baudhāyana Śulba* described above is free from any such measurement or approximation.

The applications of the theorem on the square of the diagonal to such constructions involve a subtle blend of geometric and algebraic thinking. An awareness of algebraic formulae like

$$(a \pm b)^2 = a^2 + b^2 \pm 2ab; \qquad a^2 - b^2 = (a+b)(a-b)$$

and quadratic equations, at least in a geometric form, is implicit in these constructions. We now present two more examples of exact constructions from the *Śulba* texts.

Kātyāyana Śulba-sūtra (6.7) describes the construction of a square equal in area to the sum of the areas of n equal squares, each of whose sides is of length a, say. Note that the desired square to be constructed will have area na^2, i.e., each of its sides will be of length $\sqrt{n}a$.

Figure 3 shows Kātyāyana's construction of a side of length $\sqrt{n}a$: draw a line segment BC of length $(n-1)a$ and an isoceles triangle BAC with base BC such that each of the two sides BA and AC has length $\frac{(n+1)a}{2}$. Then the altitude DA of triangle BAC has the required length $\sqrt{n}a$. For, BD $= \frac{\text{BC}}{2} = \frac{(n-1)a}{2}$ and BA $= \frac{(n+1)a}{2}$, so that $\text{DA}^2 = (\frac{(n+1)a}{2})^2 - (\frac{(n-1)a}{2})^2 = na^2$.

Figure 4 shows the *Śulba* procedure (*Baudhāyana Śulba* 1.54) to construct a square equal in area to a given rectangle ABCD of length a units and breadth b units: mark out AE of length b units along AB, complete the square DAEF and bisect the remainder EB. With the mid-point G of EB as centre, draw an arc of radius AG which intersects the extension of the line EF at I. Then the segment EI gives the desired side. Note that GE $= \frac{a-b}{2}$ and GI $=$ GA $=$ EA $+$ GE $= b + \frac{a-b}{2} = \frac{a+b}{2}$, so that $\text{EI}^2 = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2 = ab$.

[20] Note that measurements inevitably involve inaccuracy. Due to the intrinsic inaccuracy, one takes several measurements during scientific experiments.

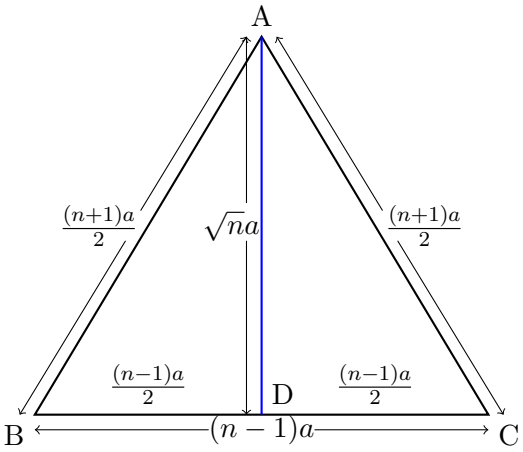


Figure 3: Kātyāyana's construction of a square whose area is n times the area of a smaller square.

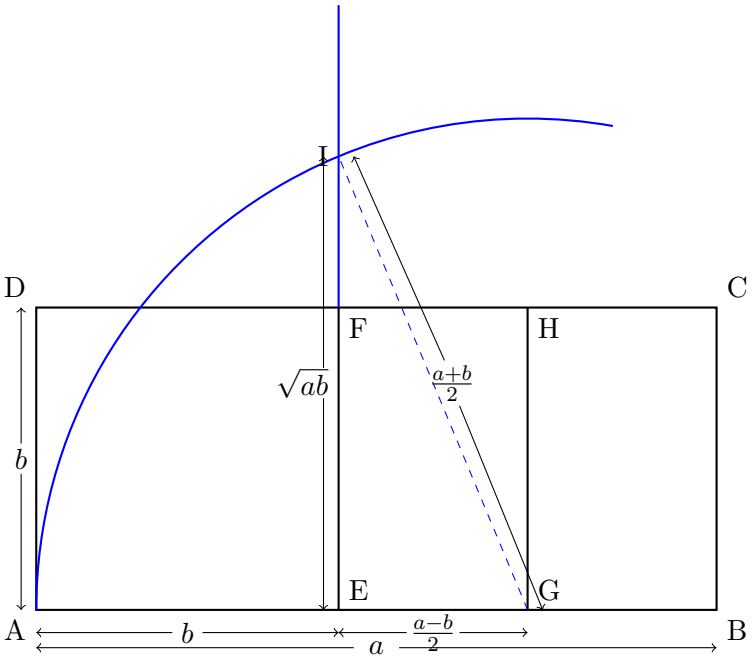


Figure 4: *Śulba* transformation of a rectangle into a square.

Seidenberg remarks[21] that this *Śulba* transformation of the rectangle into a square is in the spirit of Euclid's *Elements*:

> "entirely in the spirit of *The Elements*, Book II, indeed, I would say it's more in the spirit of Book II than Book II itself."

The above procedures: (i) give *exact* constructions of $\sqrt{n}a$ and $\sqrt{ab}$ (no measurement or approximation is involved), (ii) make ingenious applications of the theorem for the square on a diagonal, and (iii) make implicit uses of the formulae

$$na^2 = (\frac{n+1}{2})^2a^2 - (\frac{n-1}{2})^2a^2, \quad ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2.$$

In fact, the constructions may be regarded as geometric formulations of the algebraic formulae. Another method of Baudhāyana[22] for constructing progressively larger squares, from a smaller square, by adding successive gnomons, gives a geometric representation of the algebraic identity:

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2.$$

There are many such results of geometrico-algebraic flavour in the *Śulba-sūtra*s. Indeed, much of *Śulba* mathematics is algebraic in spirit and substance, and it is all the more remarkable that such accomplishments were made several centuries before the genesis of formal Algebra.[23]

The *Śulba* treatises show familiarity with the addition, multiplication and rationalization of elementary surds; the term *karaṇī* was used for surd (e.g., *dvi-karaṇī* for $\sqrt{2}$). In *Āpastamba Śulba* (5.8), one sees an implicit use of results such as[24]

$$\frac{36}{\sqrt{3}} \times \frac{1}{2} \times (\frac{24}{\sqrt{3}} + \frac{30}{\sqrt{3}}) = 324; \qquad 12\sqrt{3} \times \frac{1}{2}(8\sqrt{3} + 10\sqrt{3}) = 324.$$

The *Śulba* authors do not confine themselves to exact constructions. A noteworthy feature of *Śulba* geometry is its study of the circle and formulation of rules to construct a circle from a square and vice versa. Such constructions are inevitably approximate.[25] An interesting statement in the *Śulba-sūtra*s of Baudhāyana (1.61-1.62), Āpastamba (1.6) and Kātyāyana (2.13) is the approximation:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}.$$

[21] A. Seidenberg (a), "The Origin of Mathematics", 318.

[22] B. Datta (c), op.cit., 218.

[23] The formalisation of Algebra was to occur more than a millennium later, possibly around the time of Brahmagupta (628 CE).

[24] More details are to be found in B. Datta (c), op.cit., Chapter XVI.

[25] Modern algebra has confirmed that it is not possible to make an exact construction of a square equal in area to a circle or vice versa, using straightedge-and-compass alone.

The right hand side equals the fraction $\frac{577}{408}$ which is the best possible approximation of $\sqrt{2}$ among fractions with the same or smaller denominators.[26] In terms of decimal fractions, $\frac{577}{408}$ $(= 1.4142156\ldots)$ matches $\sqrt{2}$ $(= 1.414213\ldots)$ up to five decimal places.[27]

The *Śulba* geometry, with its theorem for the square on a diagonal, its geometrico-algebraic techniques, and its investigations on the circle, influenced post-Vedic geometry and trigonometry. Along with the decimal system, it facilitated developments in *Siddhāntic* astronomy and the emergence of concepts of calculus in India (infinite series for π and trigonometric functions, integration methods, etc.) in a geometric framework.[28]

4 Enumerative mathematics in Piṅgala's *Chandaḥ-sūtra* and other texts

The prosody text *Chandaḥ-sūtra* of Piṅgalācārya (c. 300 BCE),[29] systematises the rules for the 'metres' in Sanskrit poetry. It has 31 *sūtra*s spread over 8 chapters. We mention some of the mathematical concepts in the last 15 *sūtra*s of the eighth chapter where combinatorial tools, and what amounts to a binary representation of numbers, are used in analyzing metrical patterns.[30]

The basic entities in the discussion are *laghu* (light) and *guru* (heavy) syllables, which we denote by L and G respectively. A syllable is *guru* if it has a long vowel or (even if it is a short syllable), if what follows is a conjunct consonant, an *anusvāra*, or a *visarga*; otherwise it is *laghu*. For instance, the sequence "*sra-ṣṭu-rā-dhyā-va-ha-ti*" corresponds to G-L-G-G-L-L-L.

Prastāra

[26] A.K. Dutta (b), "Kuṭṭaka, Bhāvanā and Cakravāla", 186.

[27] The pair $(577, 408)$ satisfies the equation $x^2 - 2y^2 = 1$, a special case of an important equation investigated by Brahmagupta and other algebraists of the post-Vedic period; it is also involved in Ramanujan's prompt solution of a mathematical puzzle which had surprised P.C. Mahalanobis. See A.K. Dutta (e), "The *bhāvanā* in Mathematics", 14.

[28] For more details on *Śulba* mathematics, the reader is referred to the books by B. Datta and Sarasvati Amma, the articles by B. Datta, A. Seidenberg, S.G. Dani, A.K. Dutta, F. Staal, and the original *Śulba* texts edited by S.N. Sen and A.K. Bag, G. Thibaut, S.D. Khadilkar, and Satya Prakash and R.S. Sharma.

[29] The date of this text is uncertain; tentative estimates vary from 500 to 200 BCE, with 300 BCE being the usually preferred date.

[30] Our write-up follows closely the lecture note M.D. Srinivas (a), "Piṅgala's *Chandaḥśāstra*", and borrows extensively from it. See also R. Sridharan (a),"Sanskrit Prosody, Piṅgala *Sūtra*s and Binary Arithmetic".

In *sūtra*s 8.20-8.23, Piṅgala describes a layout (called *prastāra*) of all possible metrical patterns for a given number of syllables. The single syllable *prastāra* is a G-L pair:

1	G
2	L

In the 2-syllable *prastāra*, the first two rows are obtained by attaching a G at the right of each of the above, and the next two rows by attaching an L at the right of the above rows, as follows:

1	G	G
2	L	G
3	G	L
4	L	L

In the 3-syllable *prastāra*, the first four and the next four rows are obtained by attaching a G or an L respectively at the right of the 4 rows of the 2-syllable *prastāra*:

1	G	G	G
2	L	G	G
3	G	L	G
4	L	L	G
5	G	G	L
6	L	G	L
7	G	L	L
8	L	L	L

The procedure is likewise extended for metres with more syllables.

Set G = 0 and L = 1. Then the mirror reflection of a row in any *prastāra* is precisely the binary representation of the row-number reduced by 1. For instance, consider the 7th row in the 3-syllable *prastāra*: G L L. Its mirror reflection is L L G = 1 1 0, the binary representation of $1 \times 2^2 + 1 \times 2 + 0 \times 1 = 6$, which is the row-number 7 reduced by 1.

Saṅkhyā

*Sūtra*s 28-31 give an optimal algorithm for computing the number of metres with n syllables (called *saṅkhyā*), which we denote by S_n. As there are two possibilities (L or G) for each syllable, the *saṅkhyā* $S_n = 2 \times 2 \times 2 \ldots n$ times, i.e., $S_n = 2^n$.

The algorithm generates from n, a sequence of 2's and 0' as follows: if n is even, consider $\frac{n}{2}$ and mark '2' but if n is odd, consider $n-1$ and mark '0'; repeat the process with the obtained number ($\frac{n}{2}$ or $n-1$) and continue till you reach 0. Now

starting from 1, construct a reverse sequence[31] by doubling an obtained entry for each mark 0 and by squaring an obtained entry for each mark 2. Then the last entry in the reverse sequence is S_n.

For instance, if $n = 7$, then we get the sequence 0 2 0 2 0 from the prescribed operations $7 - 1 = 6; \frac{6}{2} = 3; 3 - 1 = 2; \frac{2}{2} = 1$ and $1 - 1 = 0$. The reversed sequence is: $1 \times 2 = 2,\ 2^2,\ 2^2 \times 2 = 2^3,\ (2^3)^2 = 2^6$ and finally $2^6 \times 2 = 2^7 = S_7$.

The same procedure is used in all later texts in India.

*Sūtra*s 8.32 and 8.33 imply the rules

$$S_1 + S_2 + S_3 + \ldots + S_n = 2 + \ldots + 2^n = 2^{n+1} - 2,$$

which include the formula for the sum of a G.P.

Naṣṭa, *Uddiṣṭa* and *Lagakriyā*

Suppose some rows in a *prastāra* are lost or '*naṣṭa*'. They can be recovered using Piṅgala's procedure to find the metrical pattern corresponding to a given row. His algorithm, called *naṣṭa*, is described in *sūtra*s 8.24-8.25. The inverse operation *uddiṣṭa*, i.e., finding the row number given the metrical pattern, is given in *sūtra*s 8.26-8.27. Both the rules are based on binary representation of a number, and the association of L with 1 and G with 0.

The *lagakriyā* process determines the number of metrical forms with r *guru*s (or *laghu*s) in a *prastāra* of metres of n syllables. This number is the binomial coefficient ${}^nC_r$ (the number of ways r objects can be chosen out of n objects).[32] The values of ${}^nC_r$'s for specific n are mentioned in various pre-300 BCE texts like *Suśrutasaṁhitā* and *Bhagavatīsūtra*.[33]

Piṅgala's *sūtra* (8.34) on *lagakriyā*, giving a cryptic hint for finding ${}^nC_r$, is too brief. Halāyudha, the tenth century commentator, explains it as giving the basic rule for the construction of a table of numbers which he refers to as the *Meru-prastāra*. The table is actually a rotated version of the well-known Pascal triangle, and is based on the recurrence relation ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.[34]

[31] The reversal reminds one of a similar 'descent'-like step in the later *kuṭṭaka* method of Āryabhaṭa. See A.K. Dutta (b), op.cit.

[32] Śrīdharācārya (750 CE), Mahāvīra (850 CE) and Hérigone (1634 CE) gave the explicit formula ${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$.

[33] In the *Bhagavatīsūtra*, there is also a recognition that there are infinities of different cardinalities (i.e., sizes) and mention is made of extending the combinatorial methods to infinite collections. This is indeed bold, considering the antiquity of the text.

[34] In the recent article by Jayant Shah, "A History of Piṅgala's Combinatorics", it is claimed

Impact of *Chandaḥ-sūtra*

The mathematical ideas in the *Chandaḥ-sūtra* had a profound impact on the development of combinatorial methods by later mathematicians like Brahmagupta (628), Śrīdharācārya (750 CE), Mahāvīra (850 CE), Bhāskarācārya (1150 CE), Nārāyaṇa Paṇdita (c. 1350 CE). In the *Prākṛta* text *Vṛtta-jāti-samuccaya* (c. 600 CE), Virahāṅka discusses the problem of computing the total number of metrical forms M_n for a given value n of the *mātrā*.[35] He shows[36] that $M_n = 1, 2, 3, 5, 8, 13, 21, \ldots$ for $n = 1, 2, 3, 4, 5, 6, 7, \ldots$, and satisfy the relation $M_n = M_{n-1} + M_{n-2}$. The Piṅgala-inspired Virahāṅka numbers are now known as 'Fibonacci numbers' after L. Fibonacci (c. 1200 CE).

The theory of Indian music involves sophisticated combinatorics where the *pratyaya*s (procedures) are essentially the same as in Piṅgala's seminal work.[37]

Statistical Ideas

There is evidence of familiarity with statistical ideas well before the Common Era. For instance, P.C. Mahalanobis, who laid the foundations of statistics in modern India, points out[38] that the *Arthaśāstra* of Kauṭilya[39] "contains a detailed description for the conduct of agricultural, population, and economic censuses in villages as well as in cities and towns *on a scale which is rare in any country even at the present time*" (italics ours).

5 Astronomy in India before 300 BCE

5.1 Astronomy in Vedic literature before *Vedāṅga Jyotiṣa*

There are references to astronomical phenomena and observations right from the early Vedic period. The Sun was considered as the Lord of the universe who supports the earth, controls the seasons and causes the winds. In the *Taittirīya Saṁhitā*

that the *lagakriyā* is actually implied in some other *sūtra* of Piṅgala's *Chandaḥ-sūtra* in its *Yajur* recension (8.23b).

[35]Here, *laghu* and *guru* are assigned the '*mātrā*' values 1 and 2 respectively, and the *mātrā* value of any metrical form is the sum of the *mātrā*s of each syllable.

[36]R. Sridharan (b), "*Pratyaya*s for *Mātrāvṛtta*s and Fibonacci Numbers"; M.D. Srinivas (b), "Development of Combinatorics", and references therein.

[37]Raja Sridharan, R. Sridharan, and M.D. Srinivas, "Combinatorial Methods in Indian Music: *Pratyayas* in *Saṅgītaratnākara* of Sārṅgadeva".

[38]P.C. Mahalanobis, "Why Statistics?", 196-197.

[39]Dated c. 300 BCE by most historians; see R.P. Kangle, *The Kauṭilya Arthaśāstra, Part III.* T.R. Trautman, who proposes a later date (150 CE) for the treatise, also suggests that some the ideas in the text have a pre-Mauryan genesis.

(abbreviated in this section as TS), the Moon is referred to as '*Sūryaraśmi*', "(one who shines by) Sun's light" (TS 3.4.7.1). The dependence of Moon's phases on its elongation from the Sun is implicit in a description in *Śatapatha Brāhmaṇa* (1.6.4.18-20). This text also describes the earth explicitly as a sphere: *parimaṇḍala u va ayam lokaḥ* (7.1.1.37).

In early Vedic literature, we find references to both lunar and solar months. A lunar month, being the time interval between two successive new Moons or full Moons, is a natural time-marker. Its average duration is nearly $29\frac{1}{2}$ days, i.e., there are 354 days in twelve lunar months. From early times it was recognised that one needs to add '*adhikamāsa*'s or 'intercalary months' at regular intervals to align the lunar months with the solar year. The concept of a calendar with 12 lunar months with intercalary months at suitable intervals, is followed to this day in India.

The *Aitareya Brāhmaṇa* (1.7.7) refers to a *saṁvatsara* with 360 days. TS (1.4.14) gives the names *Madhu, Mādhava, Śukra, Śuci, Nabhas, Nabhasya, Īṣa, Ūrja, Sahas, Sahasya, Tapas, Tapasya* to the twelve months of the year (possibly solar months of 30 days each); the intercalary months are named as '*Saṃsarpa*' and '*Aṃhaspati*'. The months are distributed among six seasons. T.S. Kuppanna Sastry points out[40] that though the solar year was roughly spoken of as having 360 days, it was known in Vedic times to have 365 days and a fraction. TS 7.1.10 mentions that 5 extra days were required over the *Sāvana* year of 360 days to complete the seasons, emphasizing that 4 days are too short and 6 days too long. In TS 7.2.6, an allusion to the extra 11 days over the 12 lunar months (totalling 354 days) can be seen in the reference to the *Ekādaśa-rātra* (11-day sacrifice) to complete the *ṛtu*s (seasons).

The northern and southern motions of the Sun along the ecliptic[41] are referred to in *Ṛg, Yajur* and *Atharva Veda*s as *uttarāyaṇa* and *dakṣiṇāyana* respectively. The treatises mention the equinoxes[42] (*viṣuvat* in Sanskrit) at the middle of the *ayana*s and the solstices at their beginning. These are important reference points in astronomy. It is noted that the Sun stands still at the winter and summer solstices.[43] A 5-year *yuga*-cycle is also mentioned in *Taittirīya* and *Vājasaneyī Saṁhitā*s.

[40] T.S. Kuppanna Sastry and K.V. Sarma, *Vedāṅga Jyotiṣa of Lagadha*, 11. Also see S.N. Sen, Astronomy, 75-76.

[41] Ecliptic is the apparent path of the Sun around the earth in the background of the stars. It is a circle on the celestial sphere inclined at an angle of nearly $23\frac{1}{2}^\circ$ with the celestial equator.

[42] The points of intersection of the ecliptic and the celestial equator.

[43] This is due to the fact that the declination of of the Sun has the least variation at these points, due to which the points of rising and settting of the Sun on the horizon do not vary over many days.

The ecliptic is divided into 27 equal divisions called *nakṣatra*s which are listed in TS and *Atharvaveda* as *Aśvini, Bharaṇi, ...* and *Revati*. As the Moon covers nearly $\frac{1}{27}$th part of the ecliptic per day, each day gets associated with the *nakṣatra* in which the Moon is situated. B.V. Subbarayappa has pointed out[44] that the nomenclature of some of the *nakṣatra*s had agricultural significance: e.g., the positioning of the Sun in the *nakṣatra Ārdrā* (literally "wet") heralds the onset of rains; *Puṣya* denotes the growth and nourishment of young sprouts; *Maghā*, the wealth of standing fruitful crop.

The *Ṛg* and *Atharva Veda*s recognise the five visible planets as being distinct from the stars; Jupiter and Venus are mentioned by name. Allusions to eclipses can be seen in verses mentioning darkness hiding the Sun (solar eclipse) and the Moon entering the Sun (lunar eclipse).

5.2 *Vedāṅga Jyotiṣa* and allied literature

The oldest available treatise exclusively devoted to astronomy is the *Vedāṅga Jyotiṣa* (abbreviated in this section as VJ). One of the limbs of the Vedas, this work is attributed to sage Lagadha and comes in two extant versions: the *Ṛgveda*ic (36 verses), and the *Yajurveda*ic (43 verses); their basic content is the same.

The VJ describes a definite quantitative calendrical system based on a cycle called *yuga* comprising 5 solar years each of 366 civil days. Thus, in VJ, 1 *yuga*= 1830 days. Each year has 2 *ayana*s and 6 seasons. The *yuga* of VJ, its first solar year, first lunar month, and the first *ayana* (*uttarāyaṇa*) commences when the Sun and the Moon are at the beginning of the winter solstice. After the completion of one *yuga*, the Sun and the Moon come together at the same position in the stellar background. In a *yuga*, there are 67 sidereal lunar months (the time required by the Moon to complete one revolution), and 62 ($= 67 - 5$) synodic or lunar months. Each lunar month has two *pakṣa*s or *parva*s, so that there are 124 *parva*s; each *parva* has 15 *tithi*s.[45] As there are 62 lunar months in 5 years, the VJ adds two additional or 'intercalary months' (*adhikamāsa*s) in each cycle of 5 years: one each in the third year and the fifth year.

The calendrical system of VJ is followed in many later texts, like the *Arthaśāstra* of Kauṭilya, *Śārdūlakarṇāvadāna*[46], Jaina texts like *Sūrya-prajñapti* and *Candra-*

[44] B.V. Subbarayappa, *Tradition of Astronomy in India - Jyotiḥśāstra*, 84.

[45] During each *tithi*, the angular separation between the Sun and the Moon increases by 12°.

[46] A Buddhist text of around the third century BCE which was translated into Chinese in third

prajñapti, and the *Paitāmaha-siddhānta*. The more advanced post-Vedic *Siddhānt*ic calendrical system too retains the VJ (and pre-VJ) concepts of intercalary months, *nakṣatra*s, *tithi*s, etc.. However, while the VJ calendar is based on the 'mean' (average) motion of the Sun and the Moon, the calculations in the *Siddhānt*ic calendrical system are based on the true positions ("longitudes") of the Sun and the Moon, which are not uniform.[47]

We cite another example of the influence of *Vedāṅga Jyotiṣa* in post-Vedic astronomy. In the 5-year VJ *yuga*-cycle, both the Sun and the Moon were considered to complete *integral* numbers of revolutions (5 and 67 respectively) around the earth. In *Āryabhaṭīya* (499 CE) and subsequent texts, we have the notion of a *Mahāyuga* of 43,20,000 years in which apart from the Sun and the Moon, all five visible planets along with their apsides and nodes complete integral numbers of revolutions.

Duration of day-time

The duration of the day-time (time-interval between the Sunrise and the Sunset) varies over the year, depending upon the position of the Sun on the ecliptic and the latitude of the place. VJ (*Ṛg* 22, *Yajur* 40) gives the linear formula $T = (12 + \frac{2n}{61})$ for the duration of the day-time T in *muhūrta*s, where n denotes the number of days elapsed after the winter solstice during *uttarāyaṇa* (or number of days yet to elapse before the winter solstice during *dakṣiṇāyana*). Y. Ohashi has argued that the formula is based on observations in North India at latitudes 27-29°N.[48]

Reference to solstices in VJ

Verses 6–8 of the VJ refer to the winter solstice being at the beginning of the asterism *Śraviṣṭhā* and the summer solstice at the mid-point of the *Āśleṣā*. Computations by historians of astronomy show that this corrresponds to a date for Lagadha in the range 1370 –1150 BCE. However, scholars have pointed out that there could be an inaccuracy of 500–600 years in this estimate, taking into account possible observational errors and other factors. Irrespective of the actual time period of Lagadha, the *extant version* of VJ might have been prepared at a later date: 500 BCE is the usual estimate.[49]

The use of gnomon for determining directions and finding time

century CE.

[47] See Chapters 1 and 3 in K. Ramasubramanian and M.S. Sriram, *Tantrasaṅgraha of Nīlakaṇṭha Somayājī*.

[48] Y. Ohashi, "Development of Astronomical Observation in Vedic and Post-Vedic India".

[49] For more details, see T.S. Kuppana Sastry and K.V. Sarma, op.cit. and Y. Ohashi, op.cit.

The geometric constructions in the *Śulba* texts are described with reference to the west-to-east direction *prācī* (the eastward line) and its perpendicular (the north-south line). Every Vedic fire-altar has a principal line of symmetry which is to be placed along the *prācī*. The *Kātyāyana Śulba-sūtra* (1.2) prescribes placing a gnomon (*śaṅku*) on a level ground, constructing a circle of radius equal to the height of the gnomon, and forming the *prācī* by joining the two points where the tip of the gnomon-shadow touches the circle in the forenoon and afternoon respectively.

Texts like *Artha-śāstra* (2.20.39-42), *Śardūlakarṇāvadāna*, *Candra-prajñapti* and *Atharva Jyotiṣa* give data on annual and diurnal variations of the shadow of a gnomon; the diurnal variation is related to the time of the day. From the data, Ohashi concludes that they are based on actual observations in North India.[50]

The influence of the above methods for fixing cardinal directions and finding time can be seen in *Siddhanti*c trigonometry and astronomy.

Bibliography

Bavare, B. and Divakaran, P.P., "Genesis and Early Evolution of Decimal Enumeration: Evidence from Number Names in Ṛgveda", *Indian J. History of Science*, 48(4), 2013, 535-581.

Dani S.G., "Geometry in the Śulvasūtras" in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*, Hindustan Book Agency, New Delhi, 2010, 9-37.

Danino, M., *The Lost River: On the trail of the Sarasvatī*, Penguin, 2010.

Datta, B. (a), "The Jaina School of Mathematics", *Bulletin of the Calcutta Mathematical Society*, Vol. 21(2), 1929, 115-145.

Datta, B. (b), "The scope and development of Hindu Gaṇita", *Indian Historical Quarterly*, V, 1929, 479-512.

Datta, B. (c), *The Science of the Śulba*, Calcutta University, 1932; reprint: Cosmo Pub., 1993.

Datta, B. (d), "Vedic Mathematics", in P. Ray and S.N. Sen, eds., *The Cultural Heritage of India, Vol. VI*, The Ramakrishna Mission Institute of Culture, Calcutta, 1986, 18-36.

Datta, B. and Singh A.N., *History of Hindu Mathematics, Parts I and II*, Motilal Banarasidass, 1935 and 1938; reprint: Asia Publishing House, 1962 and Bharatiya Kala Prakashan, 2004.

Divakaran, P.P., "Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics", in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*, Hindustan Book Agency, New Delhi, 2010, 287-351.

Dutta, A.K. (a), "Powers of Ten: Genesis and Mystic Significance", *Srinvantu*, Vol. 58(2), 44-52; reprint: *Mother India* (2009), LXII(11), 957-967 and Vedic Venues (2013), 2, 43-56.

[50] Y. Ohashi, op.cit.

Dutta A.K. (b), "Kuṭṭaka, Bhāvanā and Cakravāla", in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*, Hindustan Book Agency, New Delhi, 2010, 145-199.

Dutta, A.K. (c), "Decimal System in India" in H. Selin, ed., *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*, Springer, 2016, 1492-1504.

Dutta, A.K. (d), "Was there sophisticated mathematics during Vedic Age?" in Arijit Chaudhuri, ed., *An anthology of disparate thoughts at a popular level*, ISIREA, Kolkata, 2016, 91-132.

Dutta, A.K. (e), "The bhāvanā in Mathematics", *Bhāvanā*. Vol. 1(1), 2017, 13-19.

Ifrah, G., *The Universal History of Numbers*, John Wiley and Sons, 2000.

Joseph, G.G., *Indian Mathematics*, World Scientific, 2016.

Kangle, R.P., *The Kauṭilya Arthaśāstra, Parts I, II and III*, Bombay University, 1960, 1963 and 1965; reprint: Motilal Banarasidass (2010).

Kedaranatha, ed., *Chandaḥ-sūtra of Piṅgala with the commentary Mṛtasañjīvanī of Halāyudha Bhaṭṭa*, 3rd ed., Bombay, 1938.

Khadilkar, S.D., ed., *Kātyāyana Śulba Sūtra*, Vaidika Samsodhana Mandala, Pune, 1974.

Mahalanobis, P.C., "Why Statistics?", *Sankhyā* Vol. 10(3), 1950, 195-228.

Ohashi, Y., "Development of Astronomical Observation in Vedic and Post-Vedic India", *Indian J. History of Science*, Vol. 28 (3), 1993, 185-251.

Plofker, Kim, *History of Mathematics in India: From 500 BCE to 1800 CE*, Princeton, 2009; Indian reprint: Hindustan Book Agency, 2012.

Ramasubramanian, K. and Sriram, M.S., *Tantrasaṅgraha of Nīlakaṇṭha Somayājī*, Translation with mathematical notes, Hindustan Book Agency, New Delhi and Springer, 2011.

Sarasvati Amma, T.A., *Geometry in Ancient and Medieval India*, Motilal Banarasidass, Delhi (1979).

Sarma, K.V., "Indian Astronomy in the Vedic Age" in Siniruddha Dash, ed., "*Facets of Indian Astronomy*", Rashtriya Sanskrit University, Tirupati, 2009, 33-56.

Sarma, K.V., Ramasubramanian, K., Srinivas, M.D., and Sriram, M.S., *Gaṇita-Yukti-Bhāṣā of Jyeṣṭhadeva, Vol. I - Mathematics*, ed. with translation and explanatory notes, Hindustan Book Agency, New Delhi, 2008 and Springer, 2009.

Sastry, T.S. Kuppanna and Sarma, K.V., *Vedāṅga Jyotiṣa of Lagadha*, Critical edition, translation and notes, Indian National Science Academy, New Delhi, 1985.

Satya Prakash and Sharma R.S., eds., *Baudhāyana Śulba Sūtra* and *Āpastamba Śulba Sūtra*, Research Institute of Ancient Scientific Studies, New Delhi, 1968.

Seidenberg, A. (a), The Origin of Mathematics, *Archive for History of Exact Sciences* Vol. 18, 1978, 301-342.

Seidenberg, A. (b), The Geometry of Vedic Rituals, in F. Staal, ed., *Agni, The Vedic Ritual of the Fire Altar Vol. II*, Asian Humanities Press, Berkeley, 1983, 95-126; reprint: Motilal Banarasidass.

Sen, S.N., "Astronomy" in D.M. Bose, S.N. Sen and B.V. Subbarayappa, ed., *A Concise History of Science in India*, Indian National Science Academy, New Delhi, 1971, 58-135; second edition, Universities Press, Hyderabad, 2009.

Sen, S.N. and Bag, A.K., ed. with English translation and commentary, *Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana, and Mānava*, Indian National Science Academy, New Delhi, 1983; rev. edn. 2000.

Shah, Jayant, "A History of Piṅgala's Combinatorics", *Gaṇita Bhāratī*, Vol. 35, 2013, 1-54.

Shukla, K.S. and Sarma, K.V., ed. with English translation and notes, *Āryabhaṭīya of Āryabhaṭa*, Indian National Science Academy, New Delhi, 1976.

Sridharan, R. (a), "Sanskrit Prosody, Piṅgala *Sūtra*s and Binary Arithmetic", in G.G. Emch, R. Sridharan and M.D. Srinivas, eds., *Contributions to the History of Indian Mathematics*, Hindustan Book Agency, Delhi, 2005, 33-62.

Sridharan, R. (b), "*Pratyaya*s for *Mātrāvṛtta*s and Fibonacci Numbers", *Mathematics Teacher*, Vol. 42, 2006, 120-137.

Sridharan, Raja, Sridharan, R. and Srinivas, M.D., "Combinatorial Methods in Indian Music: *Pratyayas* in *Saṅgītaratnākara* of Sārṅgadeva" in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*, Hindustan Publishing Agency, Delhi, 2010, 55-112.

Srinivas, M.D. (a), "Piṅgala's *Chandaḥ-śāstra*", Lecture 5, NPTEL course on "*Mathematics in India from Vedic Period to Modern Times*", http://nptel.ac.in/courses/111101080/5, 2013.

Srinivas, M.D. (b), "Development of Combinatorics in India 1", Lecture 18, NPTEL course on "*Mathematics in India from Vedic Period to Modern Times*", http://nptel.ac.in/courses/111101080/18, 2013.

Staal, F., Greek and Vedic Geometry, *J. Indian Philosophy*, Vol. 27 (1/2), 1999, 105-127.

Subbarayappa, B.V. and Sarma, K.V., eds., *Indian Astronomy: A Source Book*, Nehru Centre, Bombay, 1985.

Subbarayappa, B.V., *The Tradition of Astronomy in India, Jyotiḥśāstra*, PHISPC, Vol. IV, part 4, Centre for Studies in Civilizations, New Delhi, 2008.

Thibaut, G., "On the Śulva-sūtras", *J. Asiatic Society of Bengal*, XLIV, 1875, 227-275.