WAS THERE SOPHISTICATED MATHEMATICS DURING VEDIC AGE?

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1. A remark of Swami Vivekananda

Before taking up the question in the Title, we shall try to understand a remark made by Swami Vivekananda in a speech "India's Gift to the World" delivered at the New York City[1] in 1895 ([29], Vol. II, p. 511):

> "... the ten numerals, the very cornerstone of all present civilization, were discovered in India, and are, in reality, Sanskrit words."

Here, Swamiji is referring to our present system of representing natural numbers that is now followed all over the world. "Numerals" usually refer to the symbols called "digits" with which we write numbers (example: 1,2,3,4,5,6,7,8,9,0 in English). In speech, numerals are expressed in words: "one", "two", "three", etc. As our present system expresses numbers with "ten numerals", it is called the "decimal system".[2]

The decimal system originated in India. It gradually replaced the Roman system that was prevalent in Europe and became the standard system by the 17th century CE. The adoption of the decimal system was one of the major factors for the commercial, mathematical and scientific renaissance in Europe. The decimal system made an enormous simplification of arithmetic, facilitated computations with large numbers in mathematics and the sciences, and influenced the algebraic thinking of mathematicians (as we shall see later). Its pivotal role in shaping modern civilization has been acknowledged by scholars and thinkers. For instance, the Australian historian A.L. Basham observed about the decimal system ([2], pp. 495–496):

> "Most of the great discoveries and inventions of which Europe is so proud would have been impossible without a developed system of mathematics, and this in turn would have been impossible if Europe had been shackled by the unwieldy system of Roman numerals. The unknown man who devised the new system was from the world's point of view, after the Buddha, the most important son of India. His achievement, though easily taken for granted, was the work of an analytic mind of the first order, and he deserves much more honour than he has so far received."

We thus understand the decimal system being the "cornerstone of all present civilization". But what is the implication of Swamiji's cryptic statement that the decimal system is "in reality, Sanskrit words"? The discussions in Sections 2–6 are an attempt to understand this phrase. For this, we need some clarity regarding the two main familiar forms of the decimal system:

(1) The (written) decimal place-value notation (decimal notation, in short).
(2) The corresponding (oral) decimal nomenclature.

2. The decimal place-value notation

When we write the number 2016 in the "decimal notation", the symbol 2 acquires the value "two thousand" by virtue of it being in the fourth "place" (position) from the right; while 1 acquires the value "ten" as it is in the second place. This is an example of the "place-value"

[1]The speech was delivered at the hall of the Long Island Historical Society (now renamed Brooklyn Historical Society) and reported in the Brooklyn Standard Union, February 27, 1895.

[2]The word "decimal" ("pertaining to ten") is derived from the Latin *decimalis* (adjective for "ten") and *decem* (ten), resembling the Sanskrit *daśama* (tenth) and *daśan* (ten).

principle which imparts to a digit d in the rth "place" from the right, the numerical "value" $d \times 10^{r-1}$. The symbol 0 acts as a place-holder to indicate that there is no positive contribution from the third place, i.e., no "hundred" in "two thousand and sixteen".

It is this decimal "place-value" principle that enables us to express any number in symbols using just ten digits. In Sanskrit, the term for digit is *aṅka* (mark) and the term for "place" is *sthāna*.

Mathematicians have expressed their admiration for the two brilliant innovations involved in the decimal notation: the idea of "place-value" and the concept of "zero". The great French mathematician Pierre-Simon Laplace writes (1814 CE):

> "It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity, the great ease which it has lent to all computations, puts our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity." (Quoted in [6], p. 19.)

The power of the place-value of zero has been beautifully highlighted by the American mathematician G.B. Halsted ([16], p. 20):

> "The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."[3]

Due to the simplicity of the decimal notation, so eloquently expressed by Laplace, children all over the world can now learn basic arithmetic at an early stage. This has enabled the dissemination (almost a proletarization) of a considerable body of scientific and technical knowledge which would have otherwise remained restricted only to a gifted few.

3. The decimal nomenclature

In the oral form of the decimal system, each number is expressed through

(i) words for the nine "numerals" ("one", "two", ..., "nine" in English) and
(ii) words for "powers of ten" ("ten", "hundred", "thousand", etc).

The words for the nine "numerals" play the role of digits, while the names for "powers of ten" (10^n) correspond to the place-values. The concept of zero as a place-holder is not required for expressing a number in words. For convenience, some additional derived words are also used: like "eleven", ..., "nineteen", "twenty", ..., ninety". Some of the derivations from the words for primary numbers are straightforward: e.g., "sixteen" for "six and ten".

Thus, in the verbal expression "two thousand and sixteen" for "2016", the word "thousand" tells us that 2 is in the fourth place from the right and there is no need for a phrase like "no hundred" or "zero hundred" (after "two thousand") corresponding to the 0 in 2016.

We then see that our oral decimal system of representing numbers is itself a fully evolved manifestation of the much-admired abstract place-value principle. Languages like English have adopted this system; but where did the principles of the oral decimal system originate?

[3]There is an allusion to verses from Shakespeare's A Midsummer Night's Dream (Act V, Scene I):
"And, as imagination bodies forth
The form of things unknown, the poet's pen
Turns them to shapes, and gives to airy nothing
A local habitation and a name."

4. Decimal number-vocabulary in the Ṛgveda

The oldest of the Vedas (and the oldest extant world literature) is the *Ṛgveda*, a book of 10,600 verses, compositions of mystic poets in remote antiquity. While ancient Indian traditions trace in the Vedas the seeds of all that was great and valuable in its culture, a verse of the *Ṛgveda* (10.14.15) pays homage to still older "seers of ancient times, our ancestors who carved the path".

Numbers are represented in decimal system (i.e., base 10) in the *Ṛgveda*, in all other Vedic treatises, and in all subsequent Indian texts. No other base occurs in ancient Indian texts, except a few instances of base 100 (or higher powers of 10). The *Ṛgveda* contains the current Sanskrit single-word terms for the nine primary numbers: *eka* (1), *dvi* (2), *tri* (3), *catur* (4), *pañca* (5), *ṣaṭ* (6), *sapta* (7), *aṣṭa* (8) and *nava* (9); the first nine multiples of ten (mostly derived from above): *daśa* (10), *viṁśati* (20), *triṁśat* (30), *catvāriṁśat* (40), *pañcāśat* (50), *ṣaṣṭi* (60), *saptati* (70), *aśīti* (80) and *navati* (90), and the first four powers of ten: *daśa* (10), *śata* (10^2), *sahasra* (10^3) and *ayuta* (10^4).

For compound numbers, the above names are combined as in our present verbal decimal terminology; e.g., "seven hundred and twenty" is expressed as *sapta śatāni viṁśatiḥ* in *Ṛgveda* (1.164.11). B. Bavare and P.P. Divakaran ([3]) have shown that the combination follows the Sanskrit grammatical rules of nominal composition, adopted from the Vedic time (at least) and formulated much later by Pāṇini.

5. Terms for "Powers of Ten" in Yajurveda

An enunciation of the concept of "powers of ten", a verbal manifestation of the abstract place-value principle, can be seen in the following verse of Medhātithi (*Vājasaneyī Saṁhitā* (17.2) of the *Śukla Yajurveda*), where numbers are being increased by taking progressively higher powers of ten:

> "imā me' agna' iṣṭakā dhenavaḥ santvekā ca daśa ca daśa ca śataṁ ca śataṁ
> ca sahasraṁ ca sahasraṁ cāyutaṁ cāyutaṁ ca niyutaṁ ca niyutaṁ
> ca prayutaṁ cārbudaṁ ca nyarbudaṁ ca samudrāśca madhyaṁ cāntaśca
> parārdhaścaitā me' agna' iṣṭakā dhenavaḥ santvamutrāmuṣmilloke"

A literal translation would be:

> "O Agni! May these Bricks be my fostering Cows[4] — (growing into) one and ten; ten and hundred; hundred and thousand; thousand and ten thousand; ten thousand and lakh; lakh and million; million and crore; crore and ten crores; ten crores and hundred crores; hundred crores and thousand crores; thousand crores and ten thousand crores; ten thousand crores and billion.[5] May these Bricks be my fostering Cows in yonder world as in this world!"

Here, Medhātithi explicitly describes a decuple sequence: *eka* (1), *daśa* (10), *śata* (100), *sahasra* (1000), *ayuta* (10000), *niyuta* (100000), *prayuta* (1000000), *arbuda* (10000000), *nyarbuda* (100000000), *samudra* (1000000000), *madhya* (10000000000), *anta* (100000000000) and *parārdha* (1000000000000). There is not only a specific one-word term for each power of ten up to a billion (1000000000000), each of these terms is perceived (and practically defined) to be 10 times the preceding term.

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[4]In Vedic hymns, the Cow is the symbol of consciousness in the form of knowledge and the wealth of cows symbolic of the richness of mental illumination (cf. [1], pp. 114; 42). The sanctified Bricks (*iṣṭakā*, i.e., that which helps attain the *iṣṭa*) are charged with, and represent, the mantras (cf. [13]).

[5]Billion means 10^{12} (million million) in England and Germany but 10^9 (thousand million) in USA and France. Here we use it for 10^{12}.

6. From terms for "powers of ten" to "place-value" notation

A momentous step had been taken by the ancient Vedic seers (or their unknown predecessors) when they coined single word-names to successive powers of ten. This was a verbal manifestation of the decimal "place-value principle". The written decimal place-value notation is simply

(i) a suppression of the place-names (i.e., the terms for powers of ten) from the verbal decimal expression of a number, along with
(ii) the replacement of word-names for numerals by the digit-symbols (one symbol for one digit) and
(iii) the use of a zero-symbol for the absent powers.

For instance, from the expression "two thousand sixteen" (i.e., "two thousand one ten and six"),

(i) drop "'thousand" and "ten",
(ii) replace "two" by "2", "one" by "1" and "six" by "6" and
(iii) put a "0" (after "2") to indicate that there is no "hundred" in the expression;

and we have the decimal place-value notation 2016.

Thus, the key to the decimal place-value system is the Vedic Sanskrit number-vocabulary. This gives a clue to Swamiji's statement (quoted at the beginning) that the "ten numerals" are "in reality, Sanskrit words"!

The importance of the decimal notation and the greatness of its two key ideas — place-value and zero ("a stroke of genius" in the words of B.L. van der Waerden[6]) have often been highlighted, as the quotes from Laplace and Halsted illustrate. But there has been an almost exclusive emphasis on the symbolic place-value notation. Especially, the fascination with zero appears to have overshadowed other mathematical aspects of the decimal system, and hindered the due recognition of

(i) the seminal role of the oral decimal system in the genesis of decimal place-value notation and
(ii) the mathematical sophistication involved in the invention of the oral decimal system.

We have already discussed (i). We now illustrate (ii) in Sections 7–9.

7. Vedic Number-Representation: Polynomial Aspect

The study of the properties of roots and coefficients of the quadratic polynomial $ax^2 + bx + c$ is the central theme in "Algebra" that we learn in high-school. Indeed, "polynomials" form the cornerstone of "Classical Algebra"[7]

A general polynomial of degree n in one variable may be represented in the following two ways:

(i) $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, e.g., $2x^3 + x + 6$;
(ii) $(a_n, a_{n-1}, \ldots, a_2, a_1, a_0)$, e.g., (2,0,1,6).

The sequence (ii) is usually used in Abstract Algebra to define the polynomial formally; the expression (i) is the more commonly used form for a polynomial. The polynomial (i) is also expressed in ascending powers of x, i.e., as $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n$.

[6] *Science Awakening*, P. Noordhoff Ltd., Groningen, Holland (1954), p. 56.

[7] The mathematician S.S. Abhyankar coined an interesting terminology: "High-School Algebra" for "Classical Algebra" and "College Algebra" for "Abstract Algebra". He used to emphasise the power of High-School Algebra. A mathematical poem of Abhyankar begins with the lines:

> "Polynomials and power series
> May they forever rule the world."

Note that the variable x is displayed in the more familiar form (i), but suppressed in form (ii). Also note that in (i), we need not write a coefficient if it is zero (e.g., the coefficient of x^2 is not written in $2x^3 + x + 6$); but in form (ii), 0 has to be put as an entry when any coefficient is zero.

Are we not reminded of the analogous representations of numbers in the decimal system in two ways (oral and symbolic):

(i) Two thousand One ten and Six (Two thousand and sixteen)
(ii) 2016

Indeed, the Vedic number-vocabulary has a correspondence with the usual polynomial representation (i) given by

(i) *daśa* (ten, 10) $\leftrightarrow x$,
(ii) *śata* (hundred, 10^2) $\leftrightarrow x^2$,
(iii) *sahasra* (thousand, 10^3) $\leftrightarrow x^3$,
(iv) *ayuta* (ten thousand, 10^4) $\leftrightarrow x^4$, etc.

However, unlike the symbols x^3, x^4 etc., the names for the powers of ten are not derived from *daśa* (10). As one can see, a single-word term like "thousand" is more convenient for oral communication than a derived term like "ten-to-the-power-three". Therefore, the Ancients chose distinct single-word terms for various powers of ten: *sahasra* for 10^3, *ayuta* for 10^4, and so on. But as shown in the quoted verse of Medhātithi, the numbers *śata* (hundred), *sahasra* (thousand), *ayuta* (ten thousand), etc., had indeed been perceived in the Vedic time as "powers of ten" (10^n).

A little reflection will show that the addition and multiplication of polynomials is analogous to the addition and multiplication in arithmetic based on the decimal system, except that some adjustment has to be made in arithmetic for "carrying over" due to the restriction that the numerals (corresponding to coefficients a_i) range from 0 to 9 only.

This analogy between the arithmetic based on the decimal representation of numbers and the algebra of polynomials is no coincidence; the former indeed had influenced the latter. This comes out in the following passage from Isaac Newton, a pioneer in the study of polynomials and power series in modern Europe. Here, Newton emphasises that the arithmetic of decimal numbers provides a fruitful model for developing the arithmetic operations on algebraic expressions in variables. Newton writes (1671):

> "Since the operations of computing in numbers and with variables are closely similar — indeed there appears to be no difference between them except in the characters by which quantities are denoted, definitely in the one case, indefinitely so in the latter — I am amazed that it has occurred to no one ... to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences. For since this doctrine in species has the same relationship to Algebra that the doctrine in decimal numbers has to Arithmetic, its operations of Addition, Subtraction, Multiplication, Division and Root-extraction may easily be learnt from the latter's ..." (Quoted in [11], p. 296.)

While Newton must have been referring to the decimal notation, his analogy also applies to the oral decimal system of the Vedic time. In fact, as we have seen, the oral decimal nomenclature is, in some sense, closer in spirit to the usual polynomial representation than even the decimal notation. This analogy should give us an idea of the mathematical depth in the "Sanskrit words" forming the Vedic number representation.

8. Mathematical Sophistication at the Genesis of Decimal System

The decimal system (both oral and written) is based on the mathematical principle that any natural number can be expressed in the (polynomial) form

$$a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10 + a_0$$

for some numbers $a_0, a_1, \ldots, a_n$ lying between 0 and 9.

The above fact is based on a recursive (repeated) application of the principle of division with remainder (usually called "division algorithm") for any pair of natural numbers a and b:

$$a = qb + r$$

for some numbers q, r with $0 \leq r < b$. In particular, for $b = 10$, we have

$$a = q \times 10 + r$$

for some r satisfying $0 \leq r \leq 9$. For instance, $201 = 20 \times 10 + 1$ and $2016 = 201 \times 10 + 6$.

As an example of the connection between the decimal system and division algorithm, we see that the decimal expansion of 2016 is obtained by applying the division algorithm thrice:
$2016 = 201 \times 10 + 6$
$= (20 \times 10 + 1) \times 10 + 6$
$= 20 \times 10^2 + 1 \times 10 + 6$
$= (2 \times 10) \times 10^2 + 1 \times 10 + 6$
$= 2 \times 10^3 + 1 \times 10 + 6.$

The division principle $a = qb + r$, which is at the basis of (even) the oral decimal system, pervades much of ancient Indian (and modern) mathematics. It is crucially involved in (apart from the decimal system itself) the following algorithms which were known in India by the time of Āryabhaṭa (499 CE):

(i) Long division; extraction of square root and cube root.
(ii) Finding the GCD d of a, b (which also occurs in Euclid's Elements).
(iii) Finding integers x, y such that $d = ax - by$.

The mathematical sophistication of the verbal (Vedic) decimal system can be glimpsed from the fact that its invention required the realisation and application of several abstract principles: the concept of powers of ten (a manifestation of the abstract place-value principle and the forerunner of the decimal place-value notation), division algorithm, recursive methods, and polynomial-like expansion.

For a further understanding of its mathematical significance, we shall now discuss some examples of the impact of the decimal system in the history of Indian (and world) mathematics. These discussions apply to both the oral and the place-value notation forms of the decimal system; and, in any case, the latter is a derivative of the former.

9. Impact of the Decimal System

The decimal system is largely responsible for the excellence attained by Indian mathematicians in arithmetic, astronomy and algebra. We give a few illustrations.

First, the polynomial-like decimal expansion of numbers is the basis for the efficient methods of addition, subtraction, multiplication, division, finding square root and cube root, that were developed in India at an early stage. For instance, the mutiplication algorithm involves a realisation and skilful application of the "distributive property"[8] of (ring of) integers, and the polynomial-like expansion of integers in the decimal system (plus adjustments for "carrying over"). Consider the multiplication of 19 $(= 1 \times 10 + 9)$ with 12 $(= 1 \times 10 + 2)$:

	19	$\longleftrightarrow$	$1 \times 10 + 9$		
$\times$	12	$\longleftrightarrow$	$1 \times 10 + 2$		
	38	$\longleftrightarrow$	$(1 \times 10) \times 2 + 9 \times 2$	$=$	$2 \times 10 + 10 + 8$
	190	$\longleftrightarrow$	$(1 \times 10) \times (1 \times 10) + 9 \times (1 \times 10)$	$=$	$1 \times 100 + 9 \times 10$
	228	$\longleftrightarrow$	$1 \times 100 + (3 + 9) \times 10 + 8$	$=$	$1 \times 100 + 100 + 2 \times 10 + 8$

[8]The rules: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

One performs separately $(1\times10+9)\times2$ and $(1\times10+9)\times(1\times10)$, to get, respectively $2\times10+10+8 = 3\times10+8$ and $1\times100+9\times10$; adding the two products, we get $1\times100+(3+9)\times10+8 = 2\times100+2\times10+8 = 228$. Thus, the fluency in basic Arithmetic, in ancient India and modern Europe, is due to the decimal expansion.

Incidental references indicate that all the fundamental operations of arithmetic were performed during the Vedic time. For instance, in a certain metaphysical context, it is mentioned in *Śatapatha Brāhmaṇa* (3.3.1.13) that when a thousand is divided into three equal parts, there is a remainder one.[9] The *Pañcaviṁśa Brāhmaṇa* (18.3) describes a list of sacrificial gifts forming a geometrical series

$$12, 24, 48, 96, 192, \ldots, 49152, 98304, 196608, 393216.$$

The *Śatapatha Brāhmaṇa* (10.5.4.7) mentions, correctly, the sum of an arithmetical progression

$$3(24+28+32+\ldots \text{ to 7 terms }) = 756.$$

An allegory in the *Śatapatha Brāhmaṇa* (10.24.2.2–17) lists the factors of 720:

$720\div2 = 360$; $720\div3 = 3\times80$; $720\div4 = 180$; $720\div5 = 144$; $720\div6 = 120$; $720\div8 = 90$; $720\div9 = 80$; $720\div10 = 72$; $720\div12 = 60$; $720\div15 = 48$; $720\div16 = 45$; $720\div18 = 40$; $720\div20 = 36$; $720\div24 = 30$.

And, in the *Baudhāyana Śulba*, there are examples of operations with fractions like

$$7\frac{1}{2}\div(\frac{1}{5})^2 = 187\frac{1}{2};\ 7\frac{1}{2}\div(\frac{1}{15}\text{ of }\frac{1}{2}) = 225;\ \sqrt{7\frac{1}{9}} = 2\frac{2}{3};\ (3-\frac{1}{3})^2+(\frac{1}{2}+\frac{10}{120})(1-\frac{1}{3}) = 7\frac{1}{2}$$

(Chapter XVI of [7] gives details and further examples from other *Śulba*s).

Though the (polynomial-type) methods for performing arithmetical computations are described only in post-Vedic treatises, the polynomial aspect of the Vedic number-representation indicates that their (i.e., the Vedic) methods too would have been akin to polynomial operations, using rules like (ten times ten is hundred), (ten times hundred is thousand), (hundred times hundred is ten-thousand) and so on, analogous to $xx = x^2, xx^2 = x^3, x^2x^2 = x^4$, etc.

Second, although it is identified as a topic in computational arithmetic, the decimal system (both in its verbal and notation forms) has a dormant algebraic character which influenced the algebraic thinking of mathematicians in ancient India and modern Europe. The various expansions central to modern Algebra and Number Theory, like the polynomial expansion, the power series expansion and the p-adic expansions[10] are conscious generalisations of the decimal expansion of natural numbers. In India, Brahmagupta had defined the algebra of polynomials in 628 CE and Mādhavācārya had investigated the power series expansions of trigonometric functions in 14th century CE — these post-Vedic geniuses had the advantage of being steeped in the decimal sytem gifted by the unknown seers of a remote past. As Newton would point out in 1671, a millennium after Brahmagupta, one can develop the operations (addition, multiplication, root extraction, etc.) with variables by imitating the methods of the arithmetic of the decimal system.

More recently, we see a great algebraist of the 20th century acknowledging the idea of decimal expansion in a technical innovation in his own research. In his landmark "Kyoto paper"[11] giving an exposition of the celebrated "Epimorphism Theorem", the late S.S. Abhyankar (1930–2012) writes about a crucial tool in the proof of the theorem:

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[9]The problem is also alluded to in the earlier *Ṛgveda* (6.69.8) and the *Taittirīya Saṁhitā* (3.2.11.2).

[10]While expressions like $a_0+a_1x+\cdots+a_nx^n$ are called polynomials, expressions with possibly infinitely many terms $a_0+a_1x+\cdots+a_nx^n+\ldots$ are called power series.

In the p-adic expansion of a number, one uses a prime p in place of 10 used in the decimal expansion.

[11]*On the semigroup of a meromorphic curve I*, Proceedings of the International Symposium on Algebraic Geometry, Kyoto 1977 (ed. M. Nagata), Kinokuniya, Tokyo, 249–414 (1978).

> "The concept of a strict system of generators of a semigroup is a generalisation of the idea of decimal expansion."

Indeed, the title of Chapter 2 of the paper (Chapter 1 being "Introduction") is "Decimal expansion" and that of Chapter 3 is "Polynomial expansion and approximate roots", each of these two chapters comprising ten pages.

Third, the decimal system enabled Indians to express large numbers effortlessly, right from the Vedic Age. (A Pali grammar treatise of Kāccāyana lists number-names up to 10^{140}, named *asaṅkhyeya*.) We mention two consequences.

Thanks to the decimal system, Indian astronomers could work with large time-frames, like cycles of 4320000 years, and thereby obtain strikingly accurate results. For instance, Āryabhaṭa estimated that the Earth rotates around its axis in 23 hours 56 minutes and 4.1 seconds, which matches the modern estimate (23 hours 56 minutes 4.09 seconds).

Again, it is due to the decimal system that Indian algebraists could venture into problems on equations whose solutions involve large numbers. One of the most glorious achievements of ancient Indian algebraists is their research on the problem of finding *integer* solutions of linear and quadratic equations, a very important theme in modern number theory. Three of the greatest feats on the theme are:

(i) The *kuṭṭaka* (pulverization) method of Āryabhaṭa (499 CE) for finding positive integers x, y satisfying an equation $ax - by = c$, where a, b, c are known integers.
(ii) A composition law of Brahmagupta (628 CE) on the solution space of the equation $Dx^2 + z = y^2$, where D is a fixed number.[12]
(iii) The *cakravāla* method of Jayadeva (within 11th century) for solving $Dx^2 + 1 = y^2$ in integers, a celebrated problem of modern mathematics.

As an illustration of the kind of large numbers that get involved in the above problems, we mention that the smallest pair of integers satisfying $61x^2 + 1 = y^2$ is $x = 226153980$, $y = 1766319049$. And this example occurs in the Algebra treatise *Bījagaṇita* (1150 CE) of Bhāskarācārya. 500 years later, that is, *soon after decimal system got standardised in Europe*, the equation $Dx^2 + 1 = y^2$ (and the specific example $61x^2 + 1 = y^2$) would be highlighted by Fermat (1657), heralding the advent of modern number theory.

Fourth, the traditional preoccupation with progressively large numbers, that was facilitated by the decimal system, created an environment that was conducive for the introduction of the infinite in Indian mathematics. Indian algebraists were comfortable with equations like $ax-by = c$ and $Dx^2 + 1 = y^2$ which have infinitely many solutions[13], and gave methods for generating all solutions. Bhāskarācārya (1150 CE) introduced an algebraic concept of infinity and also worked with the infinitesimal in the spirit of calculus and then there was the spectacular work on infinite series by Mādhavācārya (14th century).

Fifth, the Vedic number system is the first known example of recursive construction. Now, recursive principles are prominent features of some of the greatest mathematical achievements of ancient India like the *kuṭṭaka* and *cakravāla* methods, and the work of the Kerala school (details can be found in [14] and [11] respectively). The facility with recursive methods is another outcome of the decimal system.

A brief history of the decimal system is presented in [15] and a detailed history in [9] (also see [3], [8], [13] and [18]).

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[12]Brahmagupta's law was rediscovered by Euler in 18th century and generalised by Gauss (1801). In 2001, Manjul Bhargava stunned the mathematical world by his surprising simplification and extensions of Gauss's work. He was awarded the Fields Medal in 2014.

[13]under the conditions that c is divisible by the greatest common divisor of a and b, and that D is not a perfect square

10. Mathematics in *Vedāṅga* literature

The Vedas are primarily spiritual treatises. With passage of time, a need was felt in the later Vedic period to compose systematic texts on some of the technical knowledge of the Vedic era which were considered essential to the proper understanding and cultivation of the Vedic mantras. Thus emerged the six *Vedāṅga*s (literally, limbs of the Vedas): *śikṣā* (phonetics), *kalpa* (rituals), *vyākaraṇa* (grammar), *nirukta* (etymology), *chandas* (prosody), *jyotiṣa* (astronomy).

The importance of mathematics is emphasised in the *Vedāṅga Jyotiṣa* (4):

> yathā śikhā mayūrāṇāṁ nāgānāṁ maṇayo yathā
> tadvadvedāṅgaśāstrāṇāṁ gaṇitaṁ mūrdhani sthitam
>
> "As are the crests on the head of a peacock, as are the gems on the hoods of a snake, so is *gaṇita* at the top of the *śāstra*s known as the *Vedāṅga*."

The *Kalpa-sūtra*s (the portion of *Vedāṅga* dealing with sacrifical rites) include handbooks on geometry known as *Śulba-sūtra*s. Baudhāyana, Āpastamba and Kātyāyana are the respective authors of three of the most prominent *Śulba* texts. The *Śulba-sūtra* of Baudhāyana (estimated 800 BCE or earlier) is the world's oldest known mathematical text.

The *Śulba-sūtra*s give a compilation of principles in geometry that were used in designing the altars (called *vedi* or *citi*) where the Vedic sacrifices (*yajña*) were to be performed. The platforms of the altars were built with burnt bricks and mud mortar. The designs of the brick-altars were often intricate. For instance, the *śyenacit* has the shape of a falcon in flight (a symbolic representation of the aspiration of the spiritual seeker soaring upward); the *kūrmacit* is shaped as a tortoise, with extended head and legs, the *rathacakracit* as a chariot-wheel with spokes, and so on. Further, the Vedic tradition demanded that these constructions are executed with perfection — the accuracy had to be meticulous.

Though the geometry principles are explicitly stated only in the *Śulba-sūtra*s of a late Vedic period, they were known and applied from the earlier phases of the Vedic era. For, the constructions of Vedic altars (which involve knowledge of Śulba geometry) are described in detail in the *Śatapatha Brāhmaṇa* (a text much anterior to the *Śulba-sūtra*s), some are mentioned in the still earlier *Taittirīya Saṁhitā*, and the descriptions in these older treatises are same as those found in the *Śulba-sūtra*s. In fact, the *Śulba* authors emphasise that they are merely stating facts already known to the authors of the *Brāhmaṇa*s and *Saṁhitā*s. Even the *Ṛgveda Saṁhitā*, the oldest layer of the extant Vedic literature, mentions the sacrificial fire-altars (though without explicit descriptions of the constructions).

The *Śulba* texts (composed several centuries before Pythagoras (c. 540 BCE) and Euclid (c. 300 BCE)) show insights on the geometric and algebraic aspects of the properties of triangles, squares, rectangles, parallelograms, trapezia and circles. Plane geometry stands on two important pillars having applications throughout history: (i) the result popularly known as the "Pythagoras theorem" and (ii) the properties of similar figures. Both the features have a striking presence in the *Śulba* texts. We shall give only a few examples in the next two sections.

The prosody text *Chandaḥ-sūtra* of Piṅgalācārya (c. 300 BCE[14]) is another *Vedāṅga* treatise which is rich in mathematical ideas. Vedic Indians paid special and careful attention to the study of *chandas*. For, they had the perception that the metrical form has greater durability, power, intensity and force than the unmetrical.[15] We shall give examples of mathematical ideas from *Chandaḥ-sūtra* in Sections 14 and 15.

[14]The date is uncertain; tentative estimates vary from 500 to 200 BCE; with 300 BCE being the usually preferred date. Since Piṅgala has been referred to as an *anuja* of Pāṇini, a date closer to 500 BCE appears plausible.

[15]This was the reason why they recorded all important knowledge in verse form. In particular, all ancient Indian mathematical literature, beginning with the *Śulba-sūtra*s, are composed in verses!

11. The Baudhāyana-Pythagoras Theorem in Vedic Geometry

The most celebrated result of high-school Euclidean geometry, and one of its central pillars, is (what is known as) the Pythagoras Theorem. There are evidences to indicate that the result was known in several pre-Greek ancient civilisations; but the earliest *explicit statement* of the theorem occurs in the *Baudhāyana Śulba-sūtra* (1.48) in the following form:

> dīrghacaturaśrasyākṣṇayārajjuḥ pārśvamānī tiryaṅmānī ca yatpṛthagbhūte kurutastadubhayaṃ karoti
>
> "The diagonal of a rectangle produces [area] which its length and breadth produce separately."

Thus, Baudhāyana states that the square on the diagonal of a rectangle is equal (in area) to the sum of the squares on the two sides.[16] The theorem is stated in almost identical language ([7], p. 104) by Āpastamba (1.4) and Kātyāyana (2.11); in the Kātyāyana *Śulba*, there is an additional phrase *iti kṣetrajñānam* indicating the fundamental importance of the theorem in geometry. Indeed, the theorem would play a pivotal role in much of ancient Indian geometry and trigonometry.

Before stating the quoted version, Baudhāyana (1.45) states the special case:

> samacaturaśrasyākṣṇayārajjurdvistāvatīṁ bhūmiṁ karoti
>
> "The diagonal of a square produces an area twice as much."

From the references to fire-altars whose constructions require the knowledge of the Baudhāyana-Pythagoras Theorem, both B. Datta ([7]) and A. Seidenberg ([23]) have concluded that the result was known at the time of the Vedic treatises *Śatapatha Brāhmaṇa* and the still earlier *Taittirīya Saṁhitā*. Again, as the doubling of a square is necessary for the construction of certain altars which are mentioned even in the *Ṛgveda* (oldest Veda), B. Datta concludes that at least the special case of the theorem (for the square) was known by the time of the *Ṛgveda*.

Much fuss is made of the fact that *Śulba-sūtra*s do not give any formal proof of the theorem. But one needs to remember that these treatises are not announcing new mathematical discoveries; they are rather engineering manuals for the construction of fire-altars which include a compilation of requisite mathematical results and procedures which were already known for a long period of time. Thus, formal mathematical proofs are outside the scope of these brief and terse aphorismic texts.[17]

But, as several scholars like Thibaut, Bürk, Hankel, Schopenhauer and Datta have discussed, the various *Śulba* constructions clearly show that the *Śulba* authors knew *proofs* of the Pythagoras Theorem in some form (see [7], Chapter IX). As Manjul Bhargava, the Fields Medalist, aptly phrased it in a personal correspondence with the author: several verses in the *Śulba-sūtra*s make it pretty clear that the Śulba authors "knew why the theorem is true". We quote one such verse from Baudhāyana Śulba (1.50) which prescribes how to form a square equal in area to the sum of two given squares. It is stated shortly after the statement of the Baudhāyana-Pythagoras Theorem (1.48).

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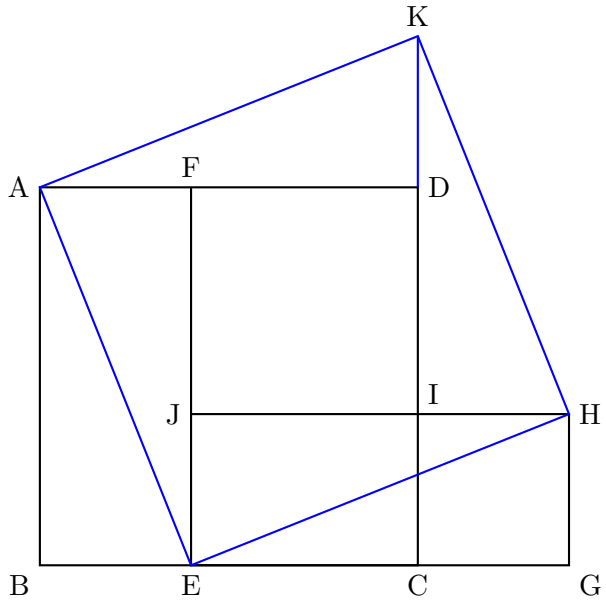
[16]It is easy to see that the above formulation is equivalent to the usual statement on the hypotenuse of a right-angled triangle.

[17]As in other disciplines, brevity is a general feature of ancient Indian mathematics treatises. The texts were meant to indicate broad hints and not complete details. There used to be emphasis on the use of a learner's own intellect for filling up the details. Most modern mathematicians feel that a researcher eventually gains more insight into his area from a terse text than from a clearly spelt-out text, from an obscure important paper than from a lucid one. The culture of brevity might have contributed to the sustained creativity displayed by ancient Indians for several centuries.

The terseness also fits with the Indian spiritual tradition. The secret meaning of the Vedic hymns were meant only for the spiritual seekers who would, after a certain stage of their development, find in the mystic verses the confirmations of their independent spiritual realisations.

nānācaturaśre samasyan kanīyasaḥ karaṇyā varṣīyaso vṛdhram ullikhet
vṛdhrasya akṣṇayārajjuḥ samastayoḥ pārśvamānī bhavati

For convenience, we supplement the literal translation with a diagram:



"To combine different squares [ABCD and ICGH], mark out the rectangle [ABEF] from the larger (square) [ABCD] with a side [BE=CG] of the smaller one [ICGH]. The diagonal [AE] of this rectangle [ABEF] is a side of the sum (of the two squares) [AEHK]."

That the area of AEHK is the sum of the areas of ABCD and ICGH becomes obvious from the described diagram, observing that the triangles ABE and EGH are being replaced, respectively, by the congruent triangles KHI and ADK.

Now, this diagram is really a demonstration of the Baudhāyana-Pythagoras Theorem! For, it shows that the square AEHK on the diagonal AE of the rectangle ABEF is the sum of the square ABCD on the side AB and the square ICGH on the side CG(=BE). Is more evidence really needed to convince oneself that the *Śulba* authors knew why the theorem holds? Indeed, there are many more of such examples (two more will occur in the next section in another context).

We also draw the reader's attention to a remark made by Seidenberg while discussing the *Śulba* verses on the trapezium ([22], p. 519):

"Many writers who refer to the Sulvasutras say that there are no proofs there. We can only suppose that these writers have not bothered to examine the work."

12. Mathematical Sophistication in Śulba constructions

The mathematical sophistication of the Vedic era can be seen from the way the Baudhāyana-Pythagoras theorem is applied to geometric constructions in the *Śulba* treatises. We shall give two examples and highlight one striking feature of these constructions which can be related to the "geometry" that we learn in high-school.

A distinction is often made in high-school between "mensuration" (which is computational) and "geometry" (where measurements are not allowed and constructions are to be performed only with a compass for drawing arcs and a ruler for drawing straight lines).[18] We shall use the

[18]The distinction is somewhat artifical, given that etymologically "geometry" is supposed to mean "earth-measurement". Seidenberg ([22], p. 520) remarks that those who first spoke of "geometry" as "earth-measurement" meant measurement of the ritual scene. He points out that the Sanskrit word "*vedi*", or its root, means "earth" and draws attention to the following passage from the *Śatapatha Brāhmaṇa* (1.2.5.7): "...By [the sacrifice] they

term "constructive geometry" for the latter. For instance, to bisect an angle, one cannot use a protractor in [constructive] geometry.

Mensuration addresses the demands of practical applications and allows numerical measurements and reasonable approximations. For instance, in mensuration, one is allowed to substitute π (the ratio of a circle's circumference to its diameter) by its convenient approximation of $\frac{22}{7}$. By contrast, constructive geometry demands mathematical rigour and exactitude. Methods involving measurements intrinsically involve an inaccuracy and have no place in constructive geometry.[19] Besides, even theoretically, one cannot find the exact measurement of certain lengths (like $\sqrt{2}$) from a standard ruler.

Constructive geometry is perhaps the earliest instance of sophisticated (as distinct from mere observational or empirical) mathematics. Its methods often involve implicit uses of algebraic results in geometric form. A. Seidenberg ([23]) postulates that constructive geometry and geometric algebra have their origin in an ancient ritual culture resembling the Vedic culture. He observes that no matter how far we go back in history, we find geometric rituals.

In the *Śulba* treatises, one sees both the constructive and the computational approaches. While there are several "exact" constructions in the *Śulba* treatises, one also sees a few instances of the use of approximations (especially in problems like squaring of a circle where exact constructions are not possible). In fact, we have already seen the constructive approach in the *Śulba* formulation of the Baudhāyana-Pythagoras Theorem (quoted earlier) as a statement on the equivalence of areas. The familiar computational version $c^2 = a^2 + b^2$ too occurs in the numerical examples in *Śulba* texts. The concern for accuracy in the building of the sacred fire-altars might have triggered the invention of the principles of constructive geometry (with its insistence on exact methods) in Vedic civilisation.

To appreciate the depth of *Śulba* geometry, recall that there were two aspects of high-school Euclidean Geometry that demanded a considerable degree of mathematical sophistication (and gave the subject an air of formidability):

(I) Constructive geometry (constructions without measurements).
(II) The logical edifice comprising axioms, theorems and proofs.

Now, while it is emphasised that the *Śulba-sūtra*s do not contain the feature (II), what is often overlooked (or suppressed) is the abundance of feature (I). Aspect (I) demands a mathematical sophistication involving a subtle blend of geometric and algebraic thinking. The *Śulba* constructions involve the Baudhāyana-Pythagoras Theorem, algebraic formulae like

$$(a \pm b)^2 = a^2 + b^2 \pm 2ab;\ a^2 - b^2 = (a+b)(a-b);$$

$$na^2 = (\frac{n+1}{2})^2 a^2 - (\frac{n-1}{2})^2 a^2;\ ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2$$

and quadratic equations. The awareness of such identities and equations, at least in a geometric form, is implicit in their work, though not formulated explicitly.

We now present two examples of exact constructions from the *Śulba* texts.

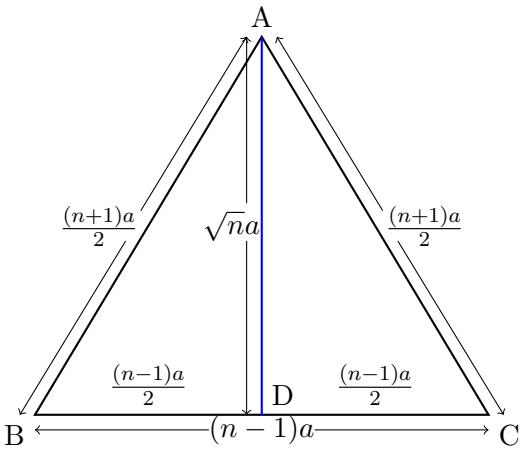
The Kātyāyana *Śulba-sūtra* (6.7) describes the following construction of a square equal in area to the sum of the areas of n squares of same size:

> yāvatpramāṇāni samacaturaśrāṇyekīkartuṁ cikīrṣedekonāni tāni bhavanti tiryak dviguṇānyekata ekādhikāni tryaśrirbhavati tasyeṣustatkaroti

obtained (*sam-vid*) this entire earth, therefore it (the sacrificial ground) is called *vedi* (the altar). For this reason they say, 'As great as the altar is, so great is the earth'; for by it (the altar) they obtained this entire (earth)...".

[19]Due to this intrinsic inaccuracy, one has to take several measurements during scientific experiments. The results are not identical and one takes the average of the observed measurements.

For clarity, we use a diagram (notation from the diagram will be added in square brackets in the translation below). We consider the side of each given square to be of length a units. Thus, the square to be constructed will have area na^2, i.e., each side of the desired square will be of length $\sqrt{n}a$.



> "To combine n squares (of equal area a^2) into one (square), construct a transverse [BC] of length $(n-1)a$; form (an isoceles) triangle [BAC] (with the transverse side [BC] as the base) such that the two sides ([BA] and [AC]) have the combined length $(n+1)a$. Then the arrow (i.e., the altitude) [DA] of the triangle [BAC] makes it (i.e., is a side of the desired square)."

Here, BD $= \frac{BC}{2} = \frac{(n-1)a}{2}$ and BA $= \frac{(n+1)a}{2}$, so that $DA^2 = (\frac{(n+1)a}{2})^2 - (\frac{(n-1)a}{2})^2 = na^2$. The procedure

(i) Gives an *exact* construction of $\sqrt{n}a$; no measurement or approximation is involved. The mensuration approach would be to draw a line of length $\sqrt{n}a$ which will be necessarily approximate if n is not a perfect square.
(ii) Makes an ingenious application of the Baudhāyana-Pythagoras Theorem.
(iii) Implicitly uses the (unstated) formula

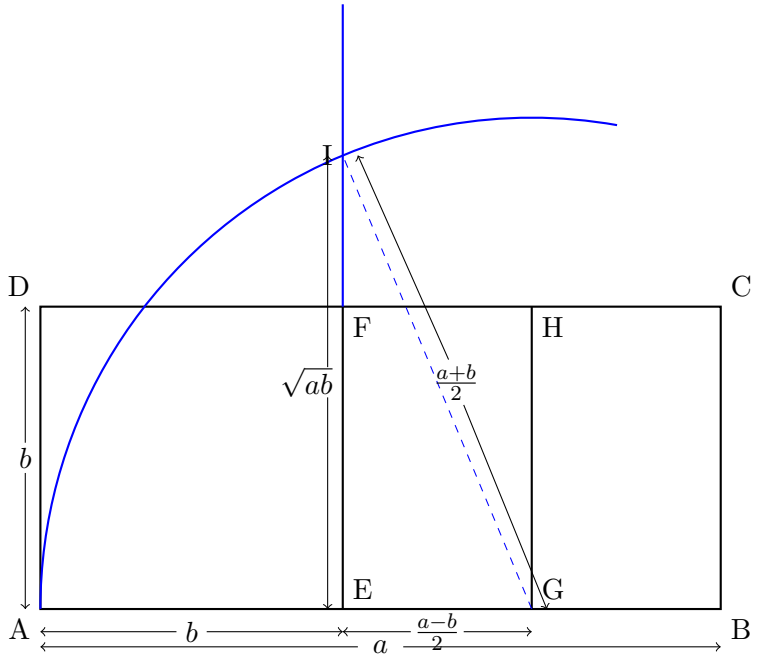
$$na^2 = (\frac{n+1}{2})^2 a^2 - (\frac{n-1}{2})^2 a^2.$$

In fact, the construction may be regarded as a geometric expression of the above algebra formula.

Next, we consider the Śulba procedure to construct a square equal in area to a given rectangle. Baudhāyana *Śulba* (1.54) states:

> dīrghacaturaśraṁ samacaturaśraṁ cikīrṣaṁstiryaṅmānīṁ karaṇīṁ kṛtvā śeṣaṁ dvedhā vibhajya viparyasyetaraccopadadhyāt khaṇḍamāvāpena tatsaṁpūrayet tasya nirhāraḥ utkaḥ

For convenience, we make use of the diagram illustrating the verse. ABCD is the given rectangle of length a units and breadth b units.



> "To transform the rectangle [ABCD] into a square: mark out [AE] of length b units along [AB] (and complete the square [DAEF]), bisect the remainder [EB] (call [G] the mid-point of [EB]). With the new point [G] (as centre), draw an arc (of radius [AG]) which intersects the extension of the line [EF] (at [I]). The segment [EI] gives the desired side."

Note that $GE = \frac{a-b}{2}$ and $GI = GA = EA+GE = b+\frac{a-b}{2} = \frac{a+b}{2}$, so that $EI^2 = (\frac{a+b}{2})^2-(\frac{a-b}{2})^2 = ab$.

The above procedure

(i) Makes an *exact* construction of $\sqrt{ab}$ from a and b — there is no measurement or approximation. The mensuration approach would be to draw a line of length $\sqrt{ab}$ which will be necessarily approximate if ab is not a perfect square.
(ii) Applies the Baudhāyana-Pythagoras Theorem.
(iii) Implicitly uses the (unstated) formula

$$ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2.$$

Again, the construction essentially gives a geometric formulation of the algebra formula. The formalisation of Algebra was to occur more than a millennium later, possibly around the time of Brahmagupta (628 CE). What is remarkable is that the Vedic savants did mathematics which was algebraic in spirit and substance (though not in form) several centuries before the genesis of formal Algebra. This makes them even greater as algebraists.[20]

The sophistication in the *Śulba* constructions is there for everyone to see. Seidenberg ([23], p. 318) remarks that the *Śulba* transformation of the rectangle into a square is in the spirit of Euclid's *Elements*:

> "entirely in the spirit of *The Elements*, Book II, indeed, I would say it's more in the spirit of Book II than Book II itself."

We mention here that the *Śulba* authors do not confine themselves to such exact constructions alone. They provide the following rational approximations to $\sqrt{2}$: $\frac{7}{5}$, $\frac{17}{12}$, $\frac{577}{408}$ (which are the 3rd,

[20]Just as one of the major achievements on which the greatness of Gauss rests is that, even before the emergence of Groups in mathematics, he did constructions amounting to imparting a group structure on an equivalence class of binary quadratic forms!

4th and 8th convergents of the simple continued fraction expansion of $\sqrt{2}$ and hence each of them is the best possible approximation among fractions with the same or smaller denominators; cf. [14], p. 186).

A striking feature of the *Śulba* geometry is its results on the circle, including construction of a circle from a square and vice versa. These constructions are inevitably approximate.[21] The pioneering work on the circle in the Vedic era appears to have a significant impact on post-Vedic mathematicians and astronomers. One is reminded of Brahmagupta's brilliant results on quadrilaterals inscribed inside a circle.

The roots of post-Vedic Indian trigonometry, with its emphasis on the sine function, can also be traced to certain features of *Śulba* geometry: the fixing of cardinal directions (East-West and North-South lines), the prominence given to the geometry of circles and, of course, the Baudhāyana-Pythagoras Theorem.[22]

Inspite of the mathematical sophistication in *Śulba* geometry, there is a tendency to dub it as "empirical" or "primitive", since these manuals do not present the subject as a logical structure built out of a set of postulates or axioms.[23] To understand this viewpoint and its flaw, we point out that the emergence of a major branch of mathematics is usually marked by three phases:

(i) empirical observations;
(ii) realisation of deep theorems and properties and an insight into their working;
(iii) axiomatic foundation and rigour.

Modern mathematicians identify stage (ii) as the birth of a subject. For instance, H. Poincaré (1854-1912) is universally acknowledged as a founder of the branch of modern mathematics called "Algebraic Topology" (which emerged during the late 19th century) since he had a deep insight into the essential ideas and results of the subject (stage (ii)), although a logical axiomatic foundation of the subject (stage (iii)) could be laid only after a subsequent effort of almost 50 years, during the 1940's. Similar is the history of Algebraic Geometry where the deep results of the Italian geometers of 19th century could be put under a rigorous axiomatic framework only towards the middle of the 20th century.

The *Śulba* geometry shows that the Vedic savants had certainly attained stage (ii) of the subject, far transcending stage (i). The following remarks by J. Dieudonné[24] about the work of Poincaré in topology may help us appreciate the nature of contributions of the pioneers of a subject, and guide us regarding what to expect (and what not to expect) from the work of the original creators of a discipline (italics in the quotations are ours):

> "... he [Poincaré] gave free rein to his imaginative powers and his extraordinary "intuition", which only very seldom led him astray; in almost every section is an original idea. *But we should not look for precise definitions, and it is often necessary to guess what he had in mind by interpreting the context. For many results, he simply gave no proof at all,* and when he endeavoured to write down a proof hardly a single argument does not raise doubts."

[21]Modern algebra has confirmed that it is not possible to make an exact construction of a square equal in area to a circle or vice versa, using ruler-and-compass alone.

[22]However, historians tend to speculate that Indian trigonometry was derived from the Greek, although the two are different in character (cf. [12]). One is reminded of the following excerpt from a report on Swami Vivekananda's sayings at the Paris Congress of the History of Religions ([29] Vol IV, p. 427): 'when the germ of every Aryan science is found in the Vedas and every step of any of those sciences can be traced with exactness from the Vedic to the present day, what is the necessity for forcing the far-fetched suggestion of the Greek influence on them? "What is the use of going to the hills in search of honey, if it is available at home?" as a Sanskrit proverb says.'

[23]The treatise of Pāṇini on Sanskrit grammar is an admirable instance of a logical edifice of a subject.

[24]Jean Dieudonné, *A History of Algebraic and Differential Topology 1900–1960*, Birkhäuser.

13. Research and Expositions on *Śulba* mathematics

Seminal research on the mathematics of the *Śulba-sutra*s was done by G. Thibaut (1875), Bibhutibhusan Datta (1932) and A. Seidenberg (1962; 1978), each of them being pioneers in their respective directions of research. Thibaut ([28]) published the *Śulba-sutra*s with his commentary and analysis. This was followed by the work of Bürk ([4]). But as Seidenberg observed in [23] (italics ours):

> "nor did he [Thibaut] formulate the *obvious conclusion, namely, that the Greeks were not the inventors of plane geometry, rather it was the Indians.* At least this was the message that the Greek scholars saw in Thibaut's paper. And they didn't like it. ... If the Indians invented plane geometry, what was to become of Greek "genius" or of the Greek "miracle"? *Most of the "refutations" were mere haughty dismissals, ... Anyway, the damage had been done and the Sulvasūtras have never taken the position in the history of mathematics that they deserve.*"

The Eurocentricism (to which Seidenberg alludes here) declined with time, but the tradition of "haughty dismissals" seems to have been taken over by a large section of Indian scholars and intellectuals (cf. Section 16).

The work of Bibhutibhusan Datta[25] ([7], [10]) contains a wealth of information and insights and is an indispensable beacon for anyone seriously interested in ancient Indian geometry. A D.Sc. in Applied Mathematics, B. Datta possessed the rare combination of professional mathematical experience with a deep knowledge of the nuances of Sanskrit literature and culture. His work on ancient Indian geometry is perhaps not so easily accessible; and the tradition of "haughty dismissals" has not helped matters.

A. Seidenberg, an eminent mathematician and historian of mathematics, has made a masterly analysis of *Śulba* mathematics in ([22], [23], [24]). Other papers which discuss specific features of *Śulba* mathematics include [17], [20], [5] and [12]. A systematic presentation of ancient Indian geometry, including Vedic geometry, occurs in the book of Sarasvati Amma ([21]). S.N. Sen and A.K. Bag have published the *Śulba-sūtra*s of Baudhāyana, Āpastamba, Kātyāyana and Mānava with commentary ([25]).

14. Binary System in Chandaḥ-sūtra

Just as any (natural) number can be represented in the decimal system using ten symbols, every number can also be represented using just two symbols 0 and 1, using the principle that any number is of the form $2^n a_n + \ldots + 2a_1 + a_0$, where each a_i is either 0 or 1.

This "binary representation" of numbers is an essential feature in the working of the digital computer since it allows the representation of any whole number using just "on" (1) and "off" (0). It was discovered in Europe by the great German mathematician and philosopher Leibniz in 1695 CE.

At least two millenniums before Leibniz, Piṅgalācārya had given mathematical rules which, in essence, correspond to the conversion of a binary representation of a number into a decimal representation and vice versa. We explain the context.

The Sanskrit language has two syllables: *laghu* (short) and *guru* (long). If one thinks of *laghu* as 1 and *guru* as 0, then the metre of a line of a Sanskrit verse corresponds to a sequence of ones and zeros and thus the binary representation of some unique natural number.

Now, Piṅgala gave a scheme by which all possible metres get labelled by unique natural numbers; i.e., he gave rules by which

(i) a distinct number (in decimal form) gets attached to a metre; and conversely,
(ii) given a number, one can recover the metre which the number represents.

[25]One of the younger brothers of Bibhutibhusan Datta (1888-1958) was Subimal Dutt (1903–92), ICS, who served as the President of the Indian Statistical Institute from 1976 to 1989.

(i) corresponds to the conversion of a binary representation to decimal, and (ii) corresponds to the conversion of a decimal representation to the binary.

One can thus see in this treatise the depth of understanding of the principles underlying place-value expansions and the mathematical sophistication in its application to the science of metres.

15. A combinatorial computation in Chandaḥ-sūtra with Zero as label

The treatise of Piṅgalācārya contains an explicit mention of *śūnya* (zero) as a distinct label along with *dvi* (two), while describing how to calculate, efficiently, the number 2^n. The context is the finding of a method to compute the total number of metres of length n, i.e., the total number of sequences of length n that can be formed using the two syllables, viz., 2^n.

Piṅgala's algorithm involves a combination of squaring and doubling. He prescribes when to square and when to double, using the principles: $2^m = (2^{\frac{m}{2}})^2$ when m is even and $2^m = 2(2^{m-1})$ when m is odd. The number n is to be halved continuously till an odd number is reached and any stage where halving is done is to be marked with label two (*dvir ardhe*). When an odd number is reached, it is to be reduced by 1 and such a stage is to be labelled by zero (*rūpe śūnyam*). Doubling has to be done corresponding to each stage marked by zero (*dviḥ śūnye*) and squaring (multiplying by itself) at each of the other stages, i.e., where a number was halved (*tāvad ardhe tad guṇitam*).

For instance, for $n = 11$, subtract 1 from 11 (put zero-mark); then divide 10 by 2 (put two); then subtract 1 from 5 (mark it by zero); then divide 4 by 2 (put two) and then 2 by 2 (put two) and finally, subtract 1 from 1 (mark it by zero). Reversing, we obtain successively 2, 2^2, $(2^2)^2$, $(2^2)^2 \times 2$, $[(2^2)^2 \times 2]^2$ and $[(2^2)^2 \times 2]^2 \times 2 (= 2^{11})$.

The reversal reminds one of a similar "descent"-like step in the later *kuṭṭaka* principle of Āryabhaṭa for solving linear indeterminate equations in integers (cf. [14]). The combinatorial sophistication in the algorithm is there for everyone to see.

Piṅgala also gave a rule to compute the total number of metres of length n which contains r *laghu* and $n - r$ *guru* syllables, i.e., to compute ${}^nC_r$ (the number of ways in which r objects can be selected out of n objects). In the 17th century, P. Herigone (1634 CE) and B. Pascal (1654 CE) described a triangular arrangement (called Pascal's triangle) for quick arithmetical computation of ${}^nC_r$ for numerical values of n and r. While explaining the terse rule of Piṅgala, his commentator Halāyudha (around 10th-11th century CE) had described the same arrangement of Herigone-Pascal (called *Meru-Prastāra* in ancient India).

More details on the mathematical gems in *Chandaḥ-sūtra* are given in the article by R. Sridharan in [27]. Piṅgala has been acknowledged by some of the later mathematically oriented prosodists like Virahāṅka[26] (600 CE). For examples of some brilliant combinatorial mathematics in ancient India inspired by Piṅgala's treatise, see the paper [26] and its bibliography.

16. Are we grateful to our Vedic Path-Makers?

To summarise some of our observations:

1. The oral decimal system used in the Vedas for expressing numbers is a highly sophisticated concept which was responsible for the excellence of Indians in arithmetic, algebra, and the analysis of infinite series.

2. The use of the Baudhāyana-Pythagoras Theorem and constructive geometry in certain *Vedāṅga* texts, especially its blending of geometric and implicit algebraic thinking, again reveal a remarkable mathematical sophistication in the Vedic civilisation. Though the geometric principles occur explicitly only in the *Śulba* treatises of a later Vedic period, the applications are made in fire-altar

[26]One of the ancient Indians who discussed the sequence of numbers named after Fibonacci, centuries before Fibonacci (c. 1200 CE).

constructions which go back to the early Vedic era. The Vedic geometry was to have impact on the geometry and trigonometry in classical Indian mathematics and astronomy.

3. Some brilliant combinatorial mathematics in ancient India was inspired by the mathematical treatment of prosody in another *Vedāṅga* treatise.

Due to the paucity of source-materials, we are not in a position to ascertain the full extent of the mathematical knowledge attained in the Vedic era. But even what has come out on the basis of limited source-materials arouses a sense of sublime wonder among sincere scholars and thinkers on ancient Indian mathematics. The impact that contemplations on Vedic geometry can have on a dedicated seeker of its history can be seen from the moving tribute paid by Bibhutibhusan Datta (emulating a verse of Kālidāsa) in the Preface of his book ([7]):

> "How great is the science which revealed itself in the *Śulba*, and how meagre is my intellect! I have aspired to cross the unconquerable ocean in a mere raft."

However, the Indian intellectual community seems to be afflicted by a strange inhibition in acknowledging the mathematical achievements and contributions of Vedic savants. Perhaps the threat of getting identified with jingoistic ideologies acts as a deterrent? The problem gets aggravated when, in the urge to achieve a "balance" between "exaggerated claims" and "complete denial", a patronising applause for post-Vedic greats like Āryabhaṭa and Brahmagupta is sought to be "balanced" by a contemptuous denial of the mathematical accomplishments during the Vedic age.

It is tragically painful to see that even a broad-minded gentle scholar like Amartya Sen, one of the very few modern Indian intellectuals who has achieved the highest possible international recognitions in his field and one who tries to be objective in his utterances, has allowed himself to get misled by the prevalent attitude of cultivated ignorance and denial. We quote a passage from his popular book "The Argumentative Indian" (2005) (pp 66–67) which has been the *raison d'être* for the present article (italics are ours):

> "As it happens, despite the richness of the Vedas in many other respects, there is *no sophisticated mathematics* in them, nor anything that can be called rigorous science. There was, however, *much* of both *in India in the first millennium CE*. These contributions were early enough in the history of mathematics and science to demand respectful attention, but the BJP-created proposed textbooks tried, with *little reason and even less evidence, to place the origin of some of these contributions in the much earlier, Vedic period.*

Are we not reminded of Seidenberg's phrase "haughty dismissals" (of Vedic contributions to sophisticated mathematics)? The contrast with the humility in Bibhuthibhusan Datta's tribute (quoted earlier) is striking. That such a passage can occur in a book by one of India's finest intellectuals 130 years after the publication of Thibaut, 73 years after the book by B. Datta and 27 years after the papers of Seidenberg, is a damning comment on the level of awareness about ancient Indian (especially the Vedic) legacy among the modern Indian elite.

Prof. Sen further says:

> "For the sake of clarity, we have to distinguish between three distinct errors that are conflated together in this invented history: ... (3) the *manifestly false affirmation* that the Vedas contain much *sophisticated mathematics* and many scientific discoveries (even though *non-partisan readers cannot find them* there)."

Then comes a bizzare remark:

> "The third claim, by the way, also has the effect of implicitly asserting that Āryabhaṭa or Brahmagupta or Varāhamihira were not original in the fifth to the seventh centuries"

Does an eulogy of an Archimedes amount to an implicit assertion that a Newton or an Euler or a Gauss were not original in the seventeenth to nineteenth centuries? The Vedic seers gave the (oral) decimal system, the post-Vedic mathematicians brought out in the open certain algebraic concepts dormant in the system; moreover, the system facilitated their original research on indeterminate

equations and their computations in astronomy. The *Śulba* authors taught the theorem on the square of the diagonal of a rectangle and results on the circle; the post-Vedic mathematicians used them to develop trigonometry and the theory of quadrilaterals inside the circle.

In the 98th Indian Science Congress (2011), Chennai, Amartya Sen proclaimed (as reported in The Statesman, Kolkata, 5.1.2011):

> 'those who are looking for the origin of Indian mathematics in the Vedas would be "completely *barking up the wrong tree*".'

We are not aware of any large-scale protest or expression of disapproval from Indian scientists and intellectuals. If we accept the quoted statements as axioms, it would then follow as a corollary that Swami Vivekananda, Bibhutibhusan Datta and most of the scholars listed in the References are "partisan" and "barking up the wrong tree"!

Among a considerable section of scientists and intellectuals, there is an anxiety that an awareness about ancient Indian (especially Vedic) achievements will lead to a pernicious "glorification" of the past, which has to be resisted by all means — by "haughty dismissals" if necessary.

But, do we not owe a sense of Gratitude to the ancients who gave us the decimal system from which emerged some of the finest mathematical ideas and techniques? Can we not take a leaf out of the book called *Ṛgveda* which gives a living demonstration of Gratitude and *Śraddhā* through the phrase in Book 10.14.15:

> idaṁ nama ṛṣibhyah: pūrvajebhyah: pūrvebhyah: pathikṛdbhyah:
>
> "Prostrations to the seers of ancient times, our ancestors who carved the path."

The Mother of Sri Aurobindo Ashram reminds us[27]:

> "The nobility of a being is measured by its capacity of gratitude."

17. A metaphor in spiritual philosophy

One of the greatest sages of India, in his conversation with one of her greatest writers, used one of her greatest mathematical concepts as a simile to emphasise one of the greatest spiritual principles. We conclude the article by recalling the metaphor. The genesis of both the spiritual and the mathematical ideas goes back to the Vedic era.

Indian spiritual philosophy regards the universe as a manifestation of the One Reality but emphasises that without the realisation of the One (the Brahman), the transient universe is an unreal vacant nought, a *śūnya*; just as many zeros form a large number if preceded by 1; but without that 1, a large collection of zeros amounts to nothing. Sri Ramakrishna mentioned this analogy in his conversation with Bankim Chandra Chattopadhyay on 6 December, 1884 (translated from Bengali by Swami Nikhilananda, [19], p. 643):

> "First realize God, then think of the creation and other things. ... If you put fifty zeros after one, you have a large sum; but erase the one and nothing remains. It is the one that makes the many. First one, then many."

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References

[1] Sri Aurobindo, *The Secret of the Veda*, Sri Aurobindo Birth Centenary Library (SABCL) Vol. 10, Sri Aurobindo Ashram, Pondicherry (1972). Originally appeared in the *Arya* in 24 instalments from August 1914 to July 1916.

[2] A.L. Basham, *The Wonder that was India*, Sidgwick and Jackson, London (1954).

[3] B. Bavare and P.P. Divakaran, Genesis and Early Evolution of Decimal Enumeration: Evidence from Number Names in Ṛgveda, *Indian Journal of History of Science*, 48(4), 535–581 (2013).

[27] Words of the Mother II, *Collected Works of the Mother* Vol 14, p. 155

[4] A.D. Bürk, Das Āpastamba Śulva-sūtra, *Zeitschrift der deutschen morgenländischen Gesellschaft*, LV, 543–591 (1901); LVI, 327–391 (1902).
[5] S.G. Dani, Geometry in the Śulvasūtras, *Studies in the History of Indian Mathematics* (ed. C.S. Seshadri), Hindustan Book Agency, New Delhi (2010).
[6] T. Dantzig, *Number: The Language of Science,* Macmillan NY (1930).
[7] B. Datta, *The Science of the Śulba*, Calcutta University (1932); Cosmo Pub. (1993).
[8] B. Datta, Vedic Mathematics, *The Cultural Heritage of India Vol VI (Science and Technology)* (ed P. Ray and S.N. Sen), The Ramakrishna Mission Institute of Culture, Calcutta (1986). Originally appeared in 1936.
[9] B. Datta and A.N. Singh, *History of Hindu Mathematics* Part I, Motilal Banarasidass (1935); reprinted by Asia Publishing House (1962) and Bharatiya Kala Prakashan (2004).
[10] B. Datta and A.N. Singh (revised by K.S. Shukla), Hindu Geometry, *Indian Journal of History of Science*, 15(2), 121–188 (1980).
[11] P.P. Divakaran, Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics, *Studies in the History of Indian Mathematics* (ed. C.S. Seshadri), Hindustan Book Agency, New Delhi (2010).
[12] P.P. Divakaran, What is Indian about Indian Mathematics? Preprint.
[13] A.K. Dutta, Powers of Ten: Genesis and Mystic Significance. *Srinvantu*, 58(2), 44–52. Reprinted in *Mother India* (2009), LXII(11), 957–967 & Vedic Venues (2013), 2, 43–56.
[14] A.K. Dutta, Kuṭṭaka, Bhāvanā and Cakravāla, *Studies in the History of Indian Mathematics* (ed. C.S. Seshadri), Hindustan Book Agency, New Delhi (2010).
[15] A.K. Dutta, Decimal System in India, *Encyclopaeida of the History of Science, Technology, and Medicine in Non-Western Cultures* (ed. H. Selin), Springer (2015).
[16] G.B. Halsted, *On the foundation and technique of Arithmetic*, Chicago: The Open Court Pub. Co. (1912).
[17] D.W. Henderson, Square Roots in the Śulba Sūtras, *Geometry at Work* (ed C. Gorini), MAA, Washington DC, 39–45 (2000).
[18] G. Ifrah, *The Universal History of Numbers*, John Wiley and Sons (2000).
[19] Swami Nikhilananda, *The Gospel of Sri Ramakrishna*, Sri Ramakrishna Math, Madras (4th edition, 1964); slightly edited translation of *Sri Ramakrishna Kathamrita* (in Bengali) by M. Gupta.
[20] J.F. Price, Applied Geometry of the Śulba Sūtras, *Geometry at Work* (ed C. Gorini), MAA, Washington DC, 46–55 (2000).
[21] T.A. Sarasvati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarasidass, Delhi (1979; 1999).
[22] A. Seidenberg, The Ritual Origin of Geometry, *Archive for History of Exact Sciences* 1, 488–527 (1962).
[23] A. Seidenberg, The Origin of Mathematics, *Archive for History of Exact Sciences* 18, 301–342 (1978).
[24] A. Seidenberg, The Geometry of Vedic Rituals, *Agni, The Vedic Ritual of the Fire Altar* Vol II (ed F. Staal), Asian Humanities Press, Berkeley, 95–126 (1983); reprinted Motilal Banarasidass.
[25] S.N. Sen and A.K. Bag (ed), *The Śulbasūtras*, Indian National Science Academy, New Delhi (1983).
[26] Raja Sridharan, R. Sridharan and M.D. Srinivas, Nārāyaṇa's Generalisation of Mātrā-vṛtta-prastāra and the Generalised Virahāṅka-Fibonacci Representation of Numbers, *Indian Journal of History of Science*, 50(2), 227–244 (2015).
[27] R. Sridharan, Sanskrit Prosody, Piṅgala Sūtras and Binary Arithmetic, *Contributions to the History of Indian Mathematics* (ed. G.G. Emch, R. Sridharan, M.D. Srinivas), Hindustan Book Agency, New Delhi (2005).
[28] G. Thibaut, On the Śulva-sūtras, *Journal of the Asiatic Society of Bengal*, XLIV, 227–275 (1875).
[29] Swami Vivekananda, *Complete Works*, Advaita Ashram, Kolkata (10th ed. 2012).

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