
Personification of वाक् in the Veda
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Calculus in Prose and Poetry: Contribution of the Kerala School

[Mādhava to Śaṅkara Vāriyar (c.1350-1550)]

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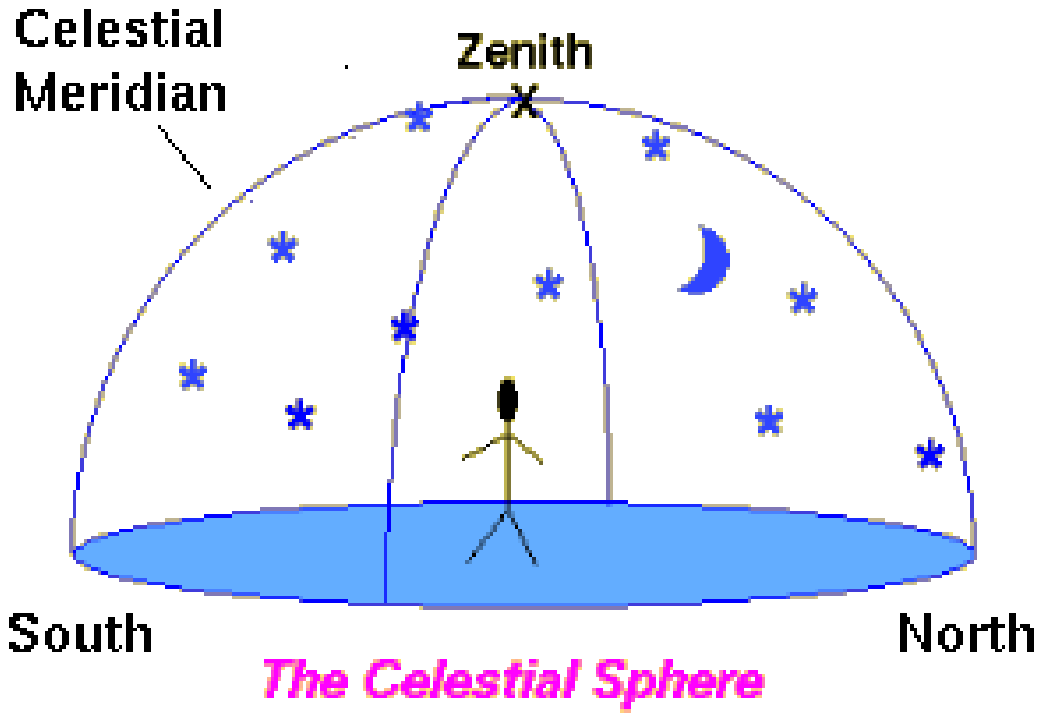
August 27, 2015
Seminar on Intellectual Traditions in Ancient India
Jain University, Bangalore

Outline

- Introduction (Discoveries, Motivation and Lineage)
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- Nīlakaṇṭha's discussion of irrationality of π
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- Estimation of sums of powers of integers 1 to *n* for large *n*
- Derivation of the Mādhava series for π
- Derivation of end-correction terms ($Antya\text{-}saṃskāra$)
- Instantaneous velocity and derivatives
- Concluding Remarks

Introduction

Celestial Sphere



The Celestial Sphere

- Great thinkers of all the civilizations – Hindu, Greek, Arabic[1], Chinese, etc. – wondered how to interpret the celestial phenomena.

[1] Nasir al-Din al-Tusi, Ibn al-Shatir, ...

Introduction

Zero and Infinity: शून्य and अनन्त

ESSENCE OF CALCULUS ≡ Use of infinitesmals/limits[2]

Greeks could not do this neat little mathematical trick. They didn't have the concept of a limit because they didn't believe in zero. The terms in the infinite series didn't have a limit or a destination; they seemed to get smaller and smaller without any particular end in sight. As a result the Greeks couldn't handle the infinite. They pondered the concept of void but rejected zero as a number, and they toyed with the concept of infinite but refused to allow infinity – numbers that are inifinitely small and infinitely large – anywhere near the realm of numbers. This is the biggest failure in the Greek Mathematics, and it is the only thing that kept them from discovering calculus. [3]

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[2] One of the passages to "limit" is by summing an infinite series.

[3] Charles Seife, *Zero:The Biography of a Dangerous Idea*, Viking, 2000; Rupa & Co. 2008.

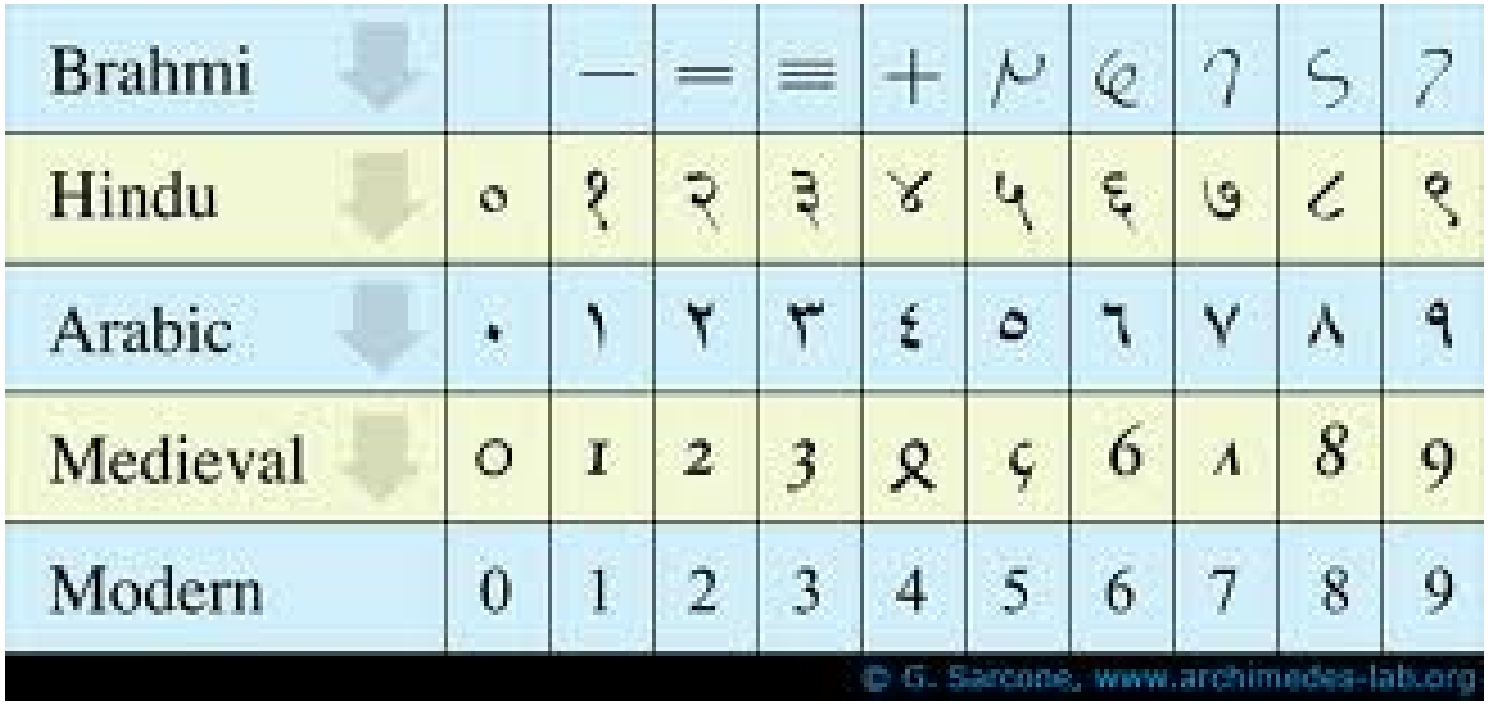
Introduction

Continuing further, Charles Seife observes:[4]

Unlike Greece, India never had the fear of the infinite or of the void. Indeed, it embraced them. . . . Indian mathematicians did more than simply accept zero. They transformed it changing its role from mere placeholder to number. The reincarnation was what gave zero its power. The roots of Indian mathematics are hidden by time. . . . Our numbers (the current system) evolved from the symbols that the Indians used; by rights they should be called Indian numerals rather than Arabic ones. . . . Unlike the Greeks the Indian did not see the squares in the square numbers or the areas of rectangles when they multiplied two different values. Instead, they saw the interplay of numerals—numbers stripped of their geometric significance. This was the birth of what we now know of *algebra*.

[4] Ibid. pp. 63–70.

Evolution of Numerals: Brahmi → Modern



- It has taken more than 18 centuries (3rd BCE – 15th CE) for the numerical notation to acquire the present form.
- The present form seems to have got adopted 'permanently' with the advent of printing press in Europe. However, there are as many as 15 different scripts used in India even today (Nagari, Bengali, Tamil (Grantha), Punjabi, Malayalam, etc.).

Ingenuity of the advent of Place value system & Zero

- Laplace[5] while describing the contribution of Indians to mathematics observes:

  The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Appolonius.

[5] A renowned French Scientist of the 18th-19th century who made phenomenal contributions to the fields of mathematics and astronomy

Description of decimal place value system

Indian philosophical literature

- In *Vyāsa-bhāṣya* on the *Yogasūtra* of Patañjali, we find an interesting description of the place value system:

  यथैका रेखा शतस्थाने शतं दशस्थाने दश एका च एकस्थाने;
  Just as the same line in the hundreds place [means] a hundred, in the tens place ten, and one in the ones place;

- In the same vein, Śaṅkara in his BSSB (2.2.17) observes:

  यथा एकोऽपि सन् देवदत्तः लोके स्वरूपं सम्बन्धिरूपं च अपेक्ष्य अनेकशब्दप्रत्ययभाग्भवति – मनुष्यः, ब्राह्मणः, श्रोत्रियः, वदान्यः, बालः, युवा, स्थविरः, पिता, पुत्रः, पौत्रः, भ्राता, जामाता इति। यथा च एकापि सती रेखा (अङ्कः) स्थानान्यत्वेन निविशमाना एक-दश-शत-सहस्रादि शब्दप्रत्ययभेदम् अनुभवति, तथा सम्बन्धिनोरेव ...

Earliest explicit use of decimal place value system

Indian mathematical and astronomical texts

- The earliest comprehensive astronomical/mathematical work that is available to us today is $\bar{A}ryabhaṭīya$ (499 CE).
- The degree of sophistication with which $\bar{\text{A}}\text{ryabhaṭa}$ has presented the number of revolutions made by the planets etc., clearly points to the fact that they had perfect knowledge of zero and the place value system.
- Moreover, his algorithms for finding square-root, cube-root etc. are also based on this.
- The system developed by $\bar{\text{A}}\text{ryabhaṭa}$ is indeed unique in the whole history of written numeration.
- Not only unique but also quite ingenious and sophisticated. Numbers of the order of 10^{16} can be represented by a single character.
- However, it was not made use of by anybody other than $\bar{\text{A}}\text{ryabhaṭa}$ — perhaps luckily as it is too complicated to read!

Signal achievements of Kerala Mathematicians

- The "Newton" series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \tag{1}$$

- The "Gregory-Leibniz"[6] series

$$Paridhi = 4 \times Vyāsa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \tag{2}$$

- The derivative of sine inverse function

$$\frac{d}{dt}\left[\sin^{-1}\left(\frac{r}{R}\sin M\right)\right] = \frac{\frac{r}{R}\cos M \, \frac{dM}{dt}}{\sqrt{1 - \left(\frac{r}{R}\sin M\right)^2}} \tag{3}$$

and many more remarkable results are found in the works of Kerala mathematicians (14th–16th cent.)

[6] The quotation marks indicate the discrepancy between the commonly employed names to these series and their historical accuracy.

Introduction

Motivation for finding the precise values of Sines and Derivatives

- Sine function ($jyā$) is ubiquitous. For instance,
  - In the computation of longitude of the planets,

$$\lambda = \lambda_0 - \sin^{-1}\left(\frac{r}{R}\sin M\right) \tag{4}$$

  - The declincation of the Sun is computed using the formula,

$$\sin\delta = \sin\epsilon \sin\lambda, \tag{5}$$

    where $\epsilon \rightarrow$ obliquity of the ecliptic and $\delta \rightarrow$ declination of the Sun.
- The time of sunrise, sunset, the computation of $lagna$, $muhūrta$ etc., heavily depend on the precise computation of $jyā$ appearing in the above relations.
- This explains the need for the computation of precise values if the $jyās$.

Sources and Lineage

- Mādhava (c.1340–1420)[7] — pioneer of the Kerala School of Mathematics.
- *Parameśvara* (c. 1380–1460) — a disciple of Mādhava, great observer and a prolofic writer.
- Nīlakaṇṭha Somayājī (c. 1444–1550) — monumental contributions *Tantrasaṅgraha* and *Āryabhaṭīya-bhāṣya*.
- Jyeṣṭhadeva, (c. 1530) — author of the celebrated *Yuktibhāṣā*.
- Śaṅkara Vāriyar (c.1500–1560) — well known for his commentaries.
- Acyuta Piṣāraṭi (c. 1550–1621) — a disciple of Jyeṣṭhadeva and a polymath.

[7] Only a couple of works of Mādhava (*Veṇvāroha* and *Sphuṭacandrāpti*) seem to be extant.

Nīlakaṇṭha's discussion of irrationality of π

- While discussing the value of π Nīlakaṇṭha observes:

परिधिव्यासयोः सङ्ख्या-सम्बन्धः प्रदर्शितः।...
आसन्नः, आसन्नतयैव अयुतद्वयसङ्ख्यविष्कम्भस्य इयं परिधिसङ्ख्या उक्ता। कुतः पुनः वास्तवीं सङ्ख्याम् उत्सृज्य आसन्नैव इहोक्ता ? उच्यते। तस्या वक्तुमशक्यत्वात्। कुतः ?

The relation between the circumference and the diameter was expressed. ...
Approximate: This value (62,832) was stated to be nearly the circumference of a circle having a diameter of 20,000. "Why then has an approximate value been mentioned here leaving behind the actual value?" It is explained [as follows]. Because it (the exact value) cannot be stated. Why?

Nīlakaṇṭha's discussion of irrationality of π

येन मानेन मीयमानो व्यासः निरवयवः स्यात्, तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात्। येन च मीयमानः परिधिः निरवयवः तेनैव मीयमानो व्यासोऽपि सावयव एव; इति एकेनैव मीयमानयोः उभयोः क्वापि न निरवयवत्वं स्यात्।

Given a certain unit of measurement ($mānа$) in terms of which the diameter ($vyāsa$) specified [is just an integer and] has no [fractional] part ($niravayava$), the same measure when employed to specify the circumference ($paridhi$) will certainly have a [fractional] part ($sāvayava$) [and cannot be just an integer]. Again if in terms of certain [other] measure the circumference has no [fractional] part, then employing the same measure the diameter will certainly have a [fractional] part [and cannot be an integer]. Thus when both [the diameter and the circumference] are measured by the same unit, they cannot both be specified [as integers] without [fractional] parts.

Nīlakaṇṭha's discussion of irrationality of π

What if I reduce the unit of measurement?

महान्तम् अध्वानं गत्वापि अल्पावयवत्वम् एव लभ्यम्।
निरवयवत्वं तु क्वापि न लभ्यम् इति भावः।
Even if you go a long way (i.e., keep on reducing the measure of the unit employed), the fractional part [in specifying one of them] will only become very small. A situation in which there will be no [fractional] part (i.e, both the diameter and circumference can be specified in terms of integers) is impossible, and this is what is the import [of the expression $\bar{a}sanna$]

What $\mathrm{N\bar{\imath}lakaṇṭha}$ is trying to explain is the incommensurability of the circumference and the diameter of a circle.
However small the unit be, the two quantities will never become commensurate – is indeed a noteworthy statement.

Sum of an infinite geometric series

Approximation for the arc of circle in terms of the $jyā$ (Rsine)

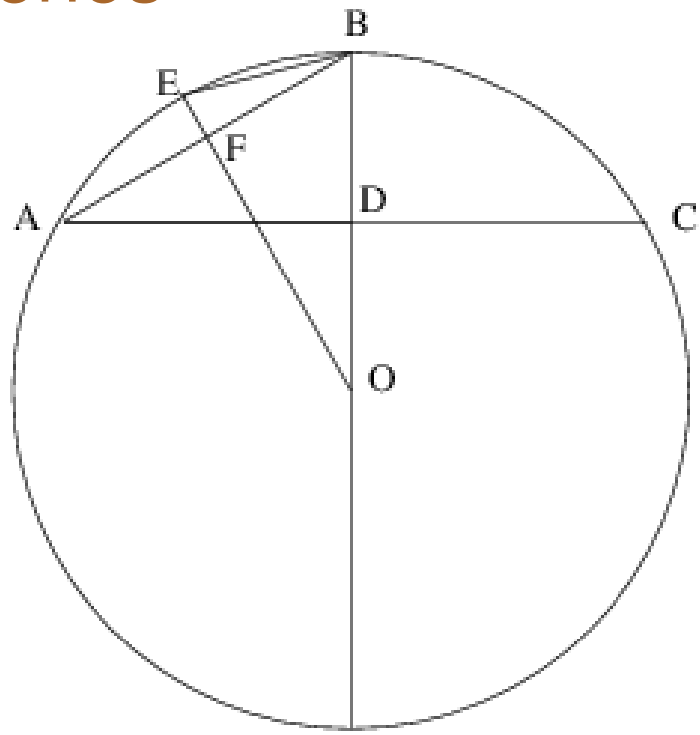
- In his $Āryabhaṭīya\text{-}bhāṣya$ – while deriving an interesting approximation for the arc of circle in terms of the $jyā$ (Rsine) and the $śara$ (Rversine) – $\mathrm{Nīlakaṇṭha}$ presents a detailed demonstration of how to sum an infinite geometric series.
- The specific geometric series that arises in the above context is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,.$$

- Here, we shall present an outline of $\mathrm{Nīlakaṇṭha}$'s argument that gives a cue to understand as to how the notion of limit was present and understood by them.

Sum of an infinite geometric series

- *AB* is $c\bar{a}pa$ (*c*) as it looks like a bow.
- *AD* is $jy\bar{a}rdha$ (*j*) as it half the string.
- *BD* is $\acute{s}ara$ (*s*) as it looks like an arrow.



The expression given by $\mathrm{N\bar{\imath}lakaṇṭha}$ is:

$$c \approx \sqrt{\left(1+\frac{1}{3}\right)\ s^2+j^2}. \tag{6}$$

सत्र्यंशादिषुवर्गात् ज्यावर्गाढ्यात् पदं धनुः प्रायः।

Sum of an infinite geometric series

The proof of (6) presented by Nīlakaṇṭha involves:

1. Repeated halving of the arc-bit, $cāpa$ c to get $c_1 \ldots c_i$.
2. Finding the corresponding semi-chords, $jyā$ (j_i) and the Rversines, $śara$ (s_i)
3. Estimating the difference between the $cāpa$ and $jyā$ at each step.

If Δ_i be the difference between the $cāpa$ and $jyā$ at the i^{th} step,

$$\Delta_i = c_i - j_i. \tag{7}$$

Here Nīlakaṇṭha observes : "as the size of the $cāpa$ decreases the difference Δ_i also decreases."

Sum of an infinite geometric series

तत्र ज्याचापयोरन्तरस्य पुनः पुनः न्यूनत्वं चापपरिमाणाल्पत्वक्रमेणेति तत्तदर्धचापानाम् अर्धज्यापरम्परा शरपरम्परा च आनीयमाना न क्वचिदपि पर्यवस्यति आनन्त्याद् विभागस्य।ततः कियन्तञ्चित् प्रदेशं गत्वा चापस्य जीवायाश्च अल्पीयस्त्वम् आपाद्य चापज्यान्तरं च शून्यप्रायं लब्ध्वा पुनरपि कल्प्यमानमन्तरम् अत्यल्पमपि कौशलात् ज्ञेयम्।

- Generating successive values of the j_is and s_is is an "unending" process as one can keep on dividing the $c\bar{a}pa$ into half *ad infinitum*.
- It would therefore be appropriate to recognize that the difference Δ_i is tending to zero and hence make an "intelligent approximation", to obtain the value of the difference between *c* and *j* approximately.

Sum of an infinite geometric series

Nīlakaṇṭha poses a very important question:

> कथं पुनः तावदेव वर्धते तावद्धर्धते च ?
> *How is it that [the sum of the series] increases only upto that [limiting value] and that certainly increases upto that [limiting value]?*

Proceeding to answer he first states the general result

$$a\left[\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\ldots\right]=\frac{a}{r-1}\,. \tag{8}$$

- Infinite Geometric Series – तुल्यच्छेदपरभागपरम्परा
- Divisor – छेद (छिद्यते अनेनेति – करणव्युत्पत्ति)

Sum of an infinite geometric series

Noting that the result is best demonstrated with $r = 4$ Nīlakaṇṭha obtains the sequence of results,

$$\begin{aligned} \frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}\,, \\ \frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,, \\ \frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}\,, \end{aligned} \tag{9}$$

and so on, which leads to the general result,

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \tag{10}$$

As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$, becomes extremely small, but never zero.

What is a Limit ?

Cauchy's (1821) definition of limit:

> *If the successive values attributed to the same variable approach indefinitely a fixed value, such that finally they differ from it by as little as one wishes, this latter is called the limit of all the others.*[8]

Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya*:

> कथं पुनः तावदेव वर्धते तावद्वर्धते च ?
> *How is it that [the sum of the series] increases only upto that [limiting value] and that certainly increases upto that [limiting value]?*

[8] Cauchy, *Cours d'Analyse*, cited by Victor J. Katz, *A History of Mathematics*, Addison Wesley Longman, New York 1998, p. 708.

Binomial series expansion

Śaṅkara Vāriyar in his *Kriyākramakarī* discusses as follows

- Consider the product $a\left(\frac{c}{b}\right)$
- Here, a is called *guṇya*, c the *guṇaka* and b the *hāra* (these are all assumed to be positive).
- If we consider the ratio $\left(\frac{c}{b}\right)$, there are two possibilities:
- **Case i:** $guṇaka > hāra$ ($c > b$). In this case we rewrite the product in the following form

$$a\left(\frac{c}{b}\right) = a + a\,\frac{(c-b)}{b}. \tag{11}$$

- **Case ii:** $guṇaka < hāra$ ($c < b$). In this case we rewrite the product as

$$a\left(\frac{c}{b}\right) = a - a\,\frac{(b-c)}{b}. \tag{12}$$

Binomial series expansion

In the expression $a\frac{(b-c)}{b}$, if we want to replace the division by b by division by c, then we have to make a subtractive correction (*śodhya-phala*) which amounts to the following equation.

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - a\frac{(b-c)}{c} \times \frac{(b-c)}{b}. \tag{13}$$

If we again replace the division by the divisor b by the multiplier c,

$$\begin{aligned} a\frac{c}{b} &= a - \left[a\frac{(b-c)}{c} - a\frac{(b-c)}{c} \times \frac{(b-c)}{c} \times \frac{c}{b}\right] \\ &= a - \left[a\frac{(b-c)}{c} - \left[a\frac{(b-c)^2}{c^2} - \left(a\frac{(b-c)^2}{c^2} \times \frac{(b-c)}{b}\right)\right]\right] \end{aligned} \tag{14}$$

The quantity $a\frac{(b-c)^2}{c^2}$ is called *dvitīya-phala* or simply *dvitīya* and the one subtracted from that is *dvitīya-śodhya-phala*.

Binomial series expansion

Thus, after taking m *śodhya-phala*-s we get

$$a\frac{c}{b} = a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^{m-1}a\left[\frac{(b-c)}{c}\right]^{m-1} + (-1)^m a\left[\frac{(b-c)}{c}\right]^{m-1}\frac{(b-c)}{b}. \quad (15)$$

एवं मुहुः फलानयने कृतेऽपि युक्तितः क्वापि न समाप्तिः। तथापि यावदपेक्षं सूक्ष्मतामापाद्य पाश्चात्यान्युपेक्ष्य फलानयनं समापनीयम्।
इहोत्तरोत्तरफलानां न्यूनत्वं तु गुणहारान्तरे गुणकारान्न्यून एव स्यात्।

- Still, if we keep including correction terms, then there is logically no end to the series of correction terms (*phala-parampar*ā).
- For achieving a given level of accuracy, we can terminate the process when the correction term becomes small enough.
- If $b - c < c$, then the successive correction terms keep decreasing.

Different approximations to π

- The *Śulba-sūtra*-s, give the value of π close to 3.088.
- $\bar{\text{A}}$ryabhaṭa (499 AD) gives an approximation which is correct to four decimal places.

  चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
  अयुतद्वयविष्कम्भस्य 'आसन्नो' वृत्तपरिणाहः॥

$$\pi \approx \frac{(100+4)\times 8 + 62000}{20000} = \frac{62832}{20000} = 3.1416$$

- Then we have the verse of *Līlāvatī*[9]

  व्यासे भनन्दाग्निहते विभक्ते खबाणसूर्यैः परिधिः सुसूक्ष्मः ।
  द्वाविंशतिघ्ने विहृतेऽथ शैलैः स्थूलोऽथवा स्याद् व्यवहारयोग्यः ॥

$$\pi = \frac{3927}{1250} = 3.1416$$ that's same as $\bar{\text{A}}$ryabhaṭa's value.

[9] *Līlāvatī* of Bhāskarācārya, verse 199.

Different approximations to π

The commentary *Kriyākramakarī* further proceeds to present more accurate values of π given by different *Ācāryas*.

माधवाचार्यः पुनः अतोप्यासन्नतमां परिधिसङ्ख्यामुक्तवान् –

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥[10]

The values of π given by the above verses are:

$$\pi = \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592$$ (correct to 11 places)

The latter one is due to Mādhava.

[10] *Vibudha=33, Netra=2, Gaja=8, Ahi=8, Hutāśana=3, Triguṇa=3, Veda=4, Bha=27, Vāraṇa=8, Bāhu=2, Nava-nikharva*=9×10^{11}. (The word *nikharva* represents 10^{11}).

Infinite series for π – as given in *Yukti-dīpikā*

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादि विषमसङ्ख्याभक्तम् ऋणं स्वं पृथक् क्रमात् कुर्यात्॥

The diameter multiplied by four and divided by unity (is found and stored). Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order (to the earlier stored result).

- *vyāse vāridhinihate* $\rightarrow$ 4 $\times$ Diameter (*vāridhi*)
- *viṣamasaṅkhyābhaktam* $\rightarrow$ Divided by odd numbers
- *triśarādi* $\rightarrow$ 3, 5, etc. (*bhūtasaṅkhyā* system)
- *ṛṇaṃ svaṃ* $\rightarrow$ to be subtracted and added [successively]

$$Paridhi = 4 \times Vyāsa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right)$$

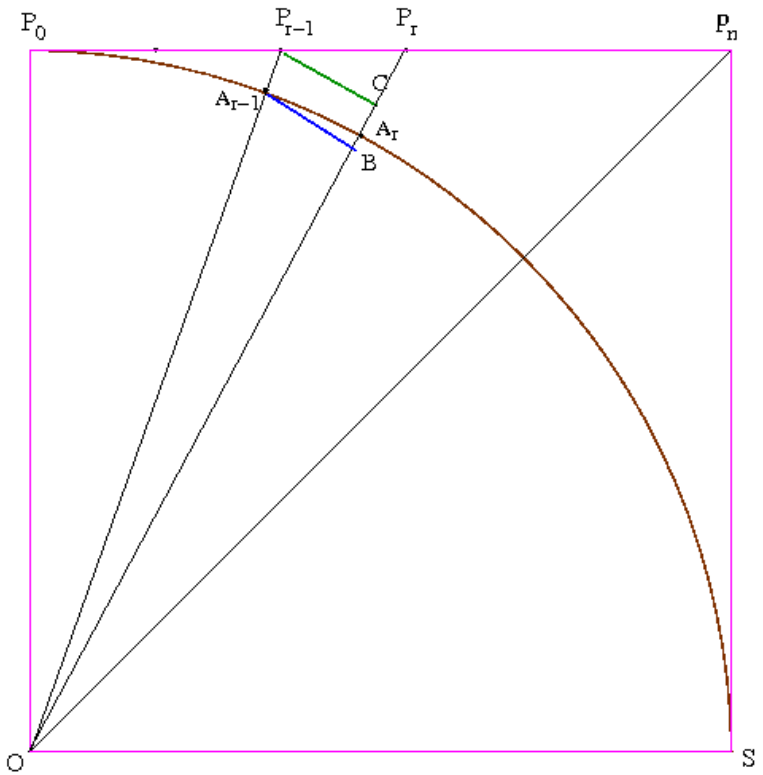
Infinite series for π

The triangles $OP_{i-1}C_i$ and $OA_{i-1}B_i$ are similar. Hence,

$$\frac{A_{i-1}B_i}{OA_{i-1}} = \frac{P_{i-1}C_i}{OP_{i-1}} \qquad (16)$$

Similarly triangles $P_{i-1}C_iP_i$ and P_0OP_i are similar. Hence,

$$\frac{P_{i-1}C_i}{P_{i-1}P_i} = \frac{OP_0}{OP_i} \qquad (17)$$



Infinite series for π

From these two relations we have,

$$\begin{aligned} A_{i-1}B_i &= \frac{OA_{i-1}.OP_0.P_{i-1}P_i}{OP_{i-1}.OP_i} \\ &= P_{i-1}P_i \times \frac{OA_{i-1}}{OP_{i-1}} \times \frac{OP_0}{OP_i} \\ &= \left(\frac{r}{n}\right) \times \frac{r}{k_{i+1}} \times \frac{r}{k_i} \\ &= \left(\frac{r}{n}\right)\left(\frac{r^2}{k_i k_{i+1}}\right). \end{aligned} \tag{18}$$

It is $\left(\frac{r}{n}\right)$ that is refered to as $khaṇḍa$ in the text. The text also notes that, when the $khaṇḍa$-s become small (or equivalently n becomes large), the Rsines can be taken as the arc-bits itself.

परिधिखण्डस्यार्धज्या $\rightarrow$ परिध्यंश
i.e., $A_{i-1}B_i \rightarrow A_{i-1}A_i$.

(local approximation by linear functions i.e., tangents/differentiation)

Infinite series for π (Error estimate)

Though the value of $\frac{1}{8}$th of the circumference has been obtained as

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0 k_1}\right) + \left(\frac{r^2}{k_1 k_2}\right) + \left(\frac{r^2}{k_2 k_3}\right) + \cdots + \left(\frac{r^2}{k_{n-1} k_n}\right)\right], \quad (19)$$

there may not be much difference in approximating it by either of the following expressions:

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right) + \left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \cdots + \left(\frac{r^2}{k_{n-1}^2}\right)\right] \quad (20)$$

$$\textit{or} \quad \frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \left(\frac{r^2}{k_3^2}\right) + \cdots + \left(\frac{r^2}{k_n^2}\right)\right] \quad (21)$$

The difference between (**??**) and (**??**) will be

$$\begin{aligned}\left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right) - \left(\frac{r^2}{k_n^2}\right)\right] &= \left(\frac{r}{n}\right)\left[1 - \left(\frac{1}{2}\right)\right] \qquad (\, k_0^2, k_n^2 = r^2, 2r^2) \\ &= \left(\frac{r}{n}\right)\left(\frac{1}{2}\right) \qquad (22)\end{aligned}$$

खण्डस्य अल्पत्ववशात् तदन्तरं शून्यप्रायमेव।

Infinite series for π

Thus we have,

$$\begin{aligned}
\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \qquad \textit{summming up/integration} \\
&= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right) + \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right)^2 - \ldots\right] \\
&= \left(\frac{r}{n}\right)[1 + 1 + \ldots + 1] \\
&\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \ldots + \left(\frac{nr}{n}\right)^2\right] \\
&\quad + \left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \ldots + \left(\frac{nr}{n}\right)^4\right] \\
&\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^6}\right)\left[\left(\frac{r}{n}\right)^6 + \left(\frac{2r}{n}\right)^6 + \ldots + \left(\frac{nr}{n}\right)^6\right] \\
&\quad + \ldots \quad . \qquad (23)
\end{aligned}$$

Infinite series for π

If we take out the powers of $bhujā\text{-}khaṇḍa\ \frac{r}{n}$, the summations involved are that of even powers of the natural numbers, namely $edādyekottara\text{-}varga\text{-}saṅkalita$, $1^2 + 2^2 + ... + n^2$, $edādyekottara\text{-}varga\text{-}varga\text{-}saṅkalita$, $1^4 + 2^4 + ... + n^4$, and so on. Kerala astronomers knew that

$$\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}. \tag{24}$$

Thus, we arrive at the result

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{25}$$

which is given in the form

$$Paridhi = 4 \times Vyāsa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots\right)$$

Summation of series (*saṅkalita*) [Integral ?]

Background

The *Āryabhaṭīya* of Āryabhaṭa has the formula for the *saṅkalita*-s

$$\begin{aligned} S_n^{(1)} &= 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \\ S_n^{(2)} &= 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \\ S_n^{(3)} &= 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \end{aligned} \tag{26}$$

From these, it is easy to estimate these sums when n is large. *Yuktibhāṣā* gives a general method of estimating the *sama-ghāta-saṅkalita*

$$S_n^{(k)} = 1^k + 2^k + \cdots + n^k, \tag{27}$$

when n is large. What it presents is a general method of estimation, which does make use of the actual value of the sum. So, the argument is repeated even for $k = 1, 2, 3$, although the result of summation is well known in these cases.

Summation of series (*saṅkalita*)

Samaghata-saṅkalita

Thus in general we have,

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \dots \\ &\approx \left(\frac{1}{k}\right) S_n^{(k)}. \end{aligned} \tag{28}$$

Rewriting the above equation we have

$$S_n^{(k)} \approx nS_n^{(k-1)} - \left(\frac{1}{k}\right) S_n^{(k)}. \tag{29}$$

(अत उत्तरोत्तरसङ्कलितानयनाय तत्तत्सङ्कलितस्य व्यासार्धगुणनम् एकैकाधिकसङ्ख्याप्त-स्वांशशोधनं च कार्यम् इति स्थितम्।)

Thus we obtain the estimate

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}. \tag{30}$$

End-correction in the infinite series for π

Need for the end-correction terms

- The series for $\frac{\pi}{4}$ is an extremely slowly convergent series.
- To obtain value of π which is accurate to 4-5 decimal places we need to consider millions of terms.
- To circumvent this problem, $\mathrm{M\bar{a}dhava}$ seems to have found an ingenious way called "$antya\text{-}saṃsk\bar{a}ra$"
- It essentially consists of –
  - Terminating the series are a particular term if you get boredom ($j\bar{a}mitay\bar{a}$).
  - Make an estimate of the remainder terms in the series
  - Apply it ($+vely/\text{-}vely$) to the value obtained by summation after termination.
- The expression provided to estimate the remainder terms is noted to be quite effective.
- Even if a consider a few terms (say 20), we are able to get π values accurate to 8-9 decimal places.

End-correction in the infinite series for π

Expression for the "remainder" terms ($Antyasaṃskāra$)

यत्सङ्ख्यात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात् ॥
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
ताभ्यामाप्तं स्वमृणे कृते घने क्षेप एव करणीयः ॥
लब्धः परिधिः सूक्ष्मः बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात् ॥

- $yatsaṅkhyayātra\ haraṇe$ $\rightarrow$ Dividing by a certain number (p)
- $nivṛttā\ hṛtistu$ $\rightarrow$ if the division is stopped
- $jāmitayā$ $\rightarrow$ being bored (due to slow-convergence)

$$\text{Remainder term} = \frac{\left(\frac{p+1}{2}\right)}{\left(\frac{p+1}{2}\right)^2 + 1}$$

- $labdhaḥ\ paridhiḥ\ sūkṣmaḥ$ $\rightarrow$ the circumference obtained would be quite accurate

End-correction in the infinite series for π

When does the end-correction give exact result ?

The discussion by Śaṅkara Vāriyar is almost in the form of a engaging dialogue between the teacher and the taught and commences with the question, how do you ensure accuracy.

कथं पुनरत्र मुहुर्विषमसङ्ख्याहरणेन लभ्यस्य परिधेः आसन्नत्वम् अन्त्यसंस्कारेण आपाद्यते? उच्यते।

तत्र तावदुक्तरूपस्संस्कारः सूक्ष्मो न वेति प्रथमं निरूपणीयम्। तदर्थं ययाकयाचिद् विषमसंख्यया हरणे कृते पृथक् संस्कारं कुर्यात्। अथ तदुत्तरविषमसंख्याहरणानन्तरं च पृथक् संस्कारं कुर्यात्। एवं कृते लब्धौ परिधी यदि तुल्यौ भवतः तर्हि संस्कारः सूक्ष्म इति निर्णीयताम्। कथम्?

How is it that you get the value close to the circumference by using antya-saṃskāra, instead of repeatedly dividing by odd numbers? This is being explained.

End-correction in the infinite series for π

When does the end-correction give exact result ?

The argument is as follows: If the correction term $\frac{1}{a_{p-2}}$ is applied after odd denominator $p-2$ (with $\frac{p-1}{2}$ is odd), then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots - \frac{1}{p-2} + \frac{1}{a_{p-2}}. \tag{31}$$

On the other hand, if the correction term $\frac{l}{a_p}$, is applied after the odd denominator p, then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots - \frac{1}{p-2} + \frac{1}{p} - \frac{1}{a_p}. \tag{32}$$

If the correction terms are exact, then both should yield the same result. That is,

$$\frac{1}{a_{p-2}} = \frac{1}{p} - \frac{1}{a_p} \qquad \text{or} \qquad \frac{1}{a_{p-2}} + \frac{1}{a_p} = \frac{1}{p}, \tag{33}$$

is the condition for the end-correction to lead to the exact result.

End-correction in the infinite series for π

Optimal choice for error-minimizaion ?

It is first observed that we cannot satisfy this condition trivially by taking $a_{p-2} = a_p = 2p$. For, the correction has to follow a uniform rule of application and thus, if $a_{p-2} = 2p$, then $a_p = 2(p+2)$;

We can, however, have both a_{p-2} and a_p close to $2p$ as possible. Hence, as first (order) estimate one tries with, "double the even number above the last odd-number divisor p", $a_p = 2(p+1)$.

But, it can be seen right away that, the condition for accuracy is not exactly satisfied. The measure of inaccuracy $(sthaulya)$ $E(p)$ is introduced, and is estimated

$$E(p) = \left(\frac{1}{a_{p-2}} + \frac{1}{a_p}\right) - \frac{1}{p}\,.$$

The objective is to find the correction denominators a_p such that the inaccuracy $E(p)$ is minimised.

End-correction in the infinite series for π

Optimal choice for error-minimizaion ?

When we set $a_p = 2(p+1)$, the inaccuracy will be

$$\begin{aligned} E(p) &= \frac{1}{(2p-2)} + \frac{1}{(2p+2)} - \frac{1}{p} \\ &= \frac{4}{(4p^3-4p)} \\ &= \frac{1}{(p^3-p)} \, . \end{aligned}$$

It can be shown that among all possible correction divisors of the type

$$a_p = 2p + m,$$

where m is an integer, the choice of $m = 2$ is optimal, as in all other cases there will arise a term proportional to p in the numerator of the inaccuracy $E(p)$.

End-correction in the infinite series for π

Optimal choice for error-minimizaion ?

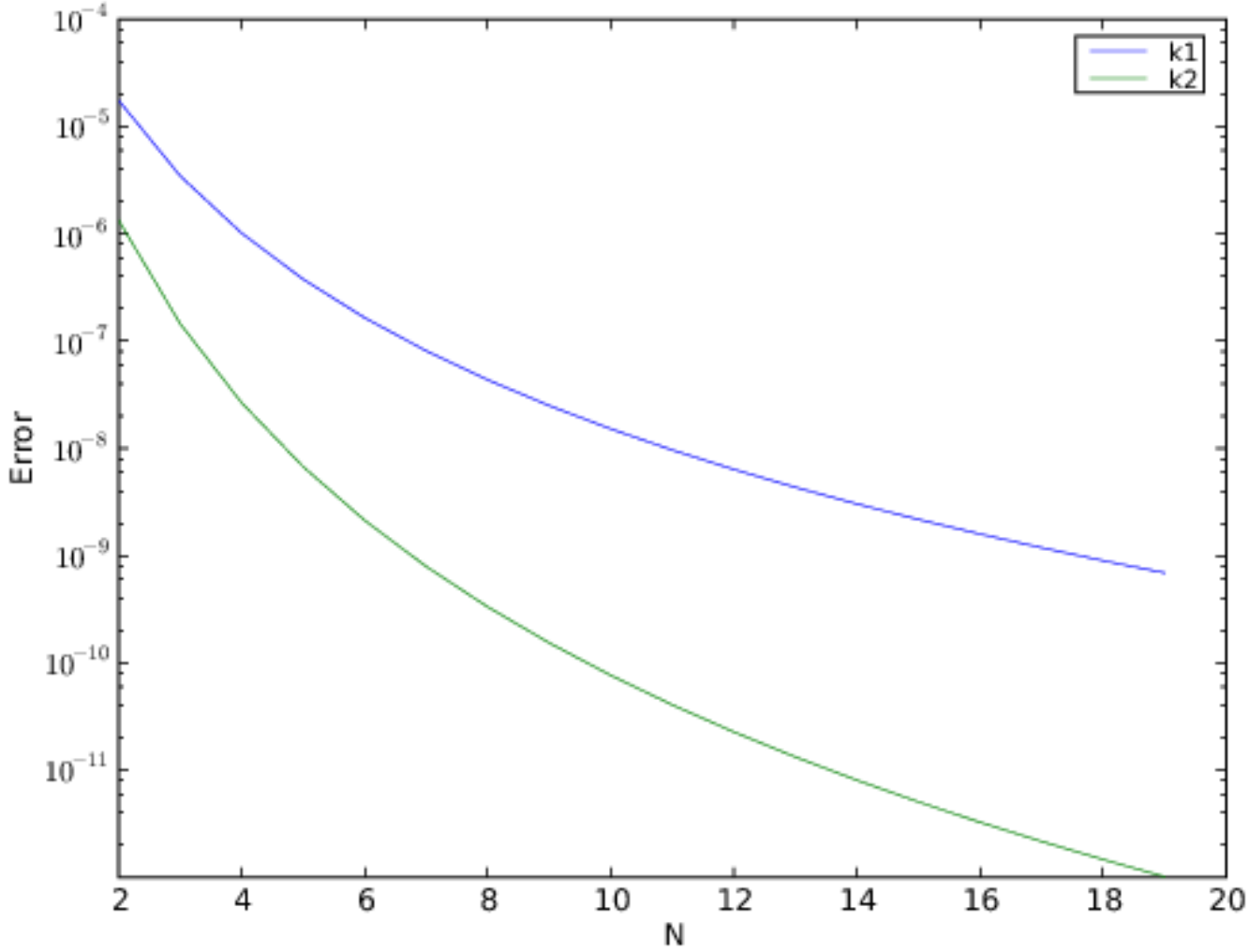
If we take the correction divisor to be $a_p = 2p + 2 + \frac{4}{(2p+2)}$, then the inaccuracy is found to be

$$\begin{aligned} E(p) &= \frac{1}{2p - 2 + \dfrac{4}{2p-2}} + \frac{1}{2p + 2 + \dfrac{4}{2p+2}} - \left(\frac{1}{p}\right) \\ &= \frac{-4}{(p^5 + 4p)} \, . \end{aligned}$$

Clearly, the $sthaulya$ with this (second order) correction divisor has improved considerably, in that it is now proportional to the inverse fifth power of the odd number.

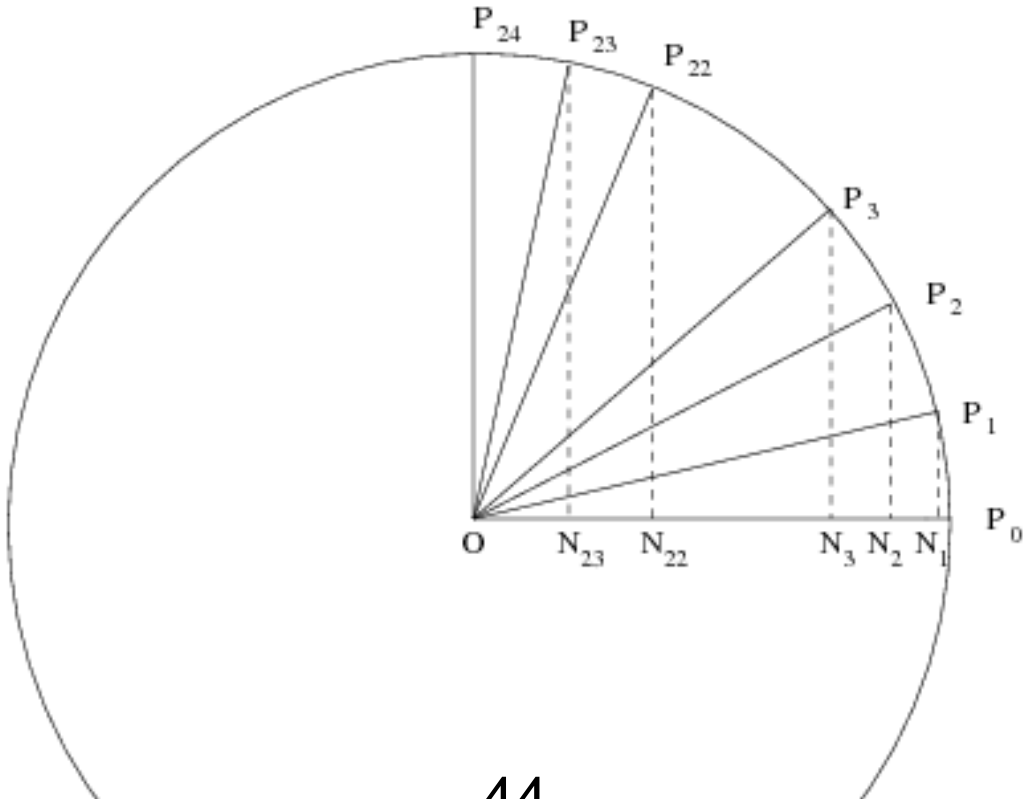
It can be shown that if we take any other correction divisor $a_p = 2p + 2 + \frac{m}{(2p+2)}$, where m is an integer, we will end up having a contribution proportional to p^2 in the numerator of the inaccuracy $E(p)$, unless $m = 4$.

Error-minimization in the evaluation of Pi



Construction of the Sine-table

- A quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^\circ 45' = 225'$.
- A procedure for finding $R \sin i\alpha, \quad i = 1, 2, \ldots 24$ is explicitly given. $P_i N_i$ are known.
- The R sines of the intermediate angles are determined by interpolation (I order or II order).



Recursion relation for the construction of sine-table

$\bar{A}ryabhaṭīya$'s algorithm for constructing of sine-table

- The content of the verse in $\bar{A}ryabhaṭīya$ translates to:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - \frac{R\sin i\alpha}{R\sin\alpha}.$$

- In fact, the values of the 24 *Rsines* themselves are explicitly noted in another verse.
- The exact recursion relation for the Rsine differences is:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - R\sin i\alpha \;\; 2(1-\cos\alpha).$$

- Approximation used by $\bar{\text{A}}\text{ryabhaṭa}$ is $2(1-\cos\alpha) = \frac{1}{225}$.
- While, $2(1-\cos\alpha) = 0.0042822$, $\frac{1}{225} = 0.00444444$.
- In the recursion relation provided by Nīlakaṇṭha we find $\frac{1}{225} \rightarrow \frac{1}{233.5} (= 0.0042827)$.

Comment on Āryabhaṭa's Method (Delambre)

Commenting upon the method of Āryabhaṭa in his monumental work Delambre[11] observes:

"The method is curious: it indicates a method of calculating the table of sines by means of their second differences. . . . The differential process has not up to now been employed except by Briggs, who himself did not know that the constant factor was the square of the chord . . . Here then is a method which the Indians possessed and which is found neither amongst the Greeks nor amongst the Arabs."[12]

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[11] ". . . an astronomer of wisdom and fortitude, able to review 130 years of astronomical observations, assess their inadequacies, and extract their value." – Prix prize citation 1789.

[12] Delambre, *Historie de l'Astronomie Ancienne*, t 1, Paris 1817, p.457; cited from B. Datta and A. N. Singh, Hindu Trigonometry, IJHS 18, 1983, p.77.

Infinite series for the sine function

- The verses giving the ∞ series for the sine function is[13] –

  निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
  हरेत् सम्मूलयुग्वर्गैः त्रिज्यावर्गहतैः क्रमात् ॥
  चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत् ।
  जीवास्त्यै, सङ्ग्रहोऽस्यैव विद्वानित्यादिना कृतः ॥

- $N_0 = R\theta$ $\quad D_0 = 1$
- $N_1 = R\theta \times (R\theta)^2$ $\quad N_{i+1} = N_i \times (R\theta)^2$
- $D_1 = R^2(2 + 2^2)$ $\quad D_i = D_{i-1} \times R^2(2i + (2i)^2)$
- जीवा $= \frac{N_0}{D_0} - [\frac{N_1}{D_1} - (\frac{N_2}{D_2} - \{\frac{N_3}{D_3} - \dots\})]$
- जीवास्त्यै = For obtaining the $j\bar{\imath}va$ (Rsine)

[13] $Yuktid\bar{\imath}pik\bar{a}$ (16th cent) and attributed to Mādhava (14th cent. AD).

Infinite series for the sine function

- Expressing the series using modern notation as described as described in the above verse –

$$Jīvā = R\theta - \frac{R\theta \times (R\theta)^2}{R^2(2+2^2)} + \frac{R\theta \times (R\theta)^2 \times (R\theta)^2}{R^2(2+2^2)\ R^2(4+4^2)} - \dots$$

- Simplifying the above we have –

$$Jīvā = R\theta - \frac{(R\theta)^3}{R^2 \times 6} + \frac{(R\theta)^5}{R^4 \times 6 \times 20} - \frac{(R\theta)^7}{R^6 \times 6 \times 20 \times 42} + \dots$$

- Further simplifying –

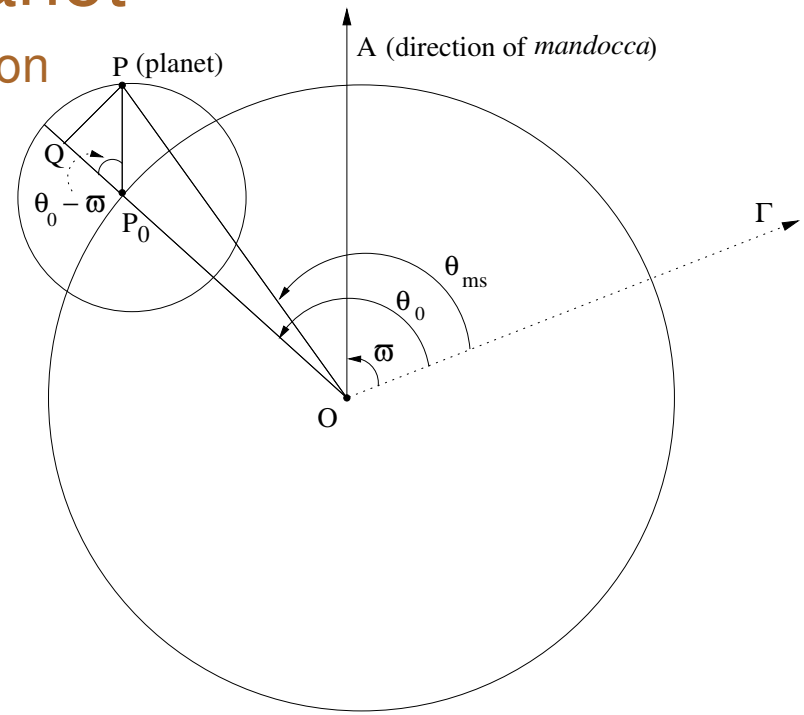
$$Jīvā = R\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) = R\sin\theta$$

- Thus the given expression $\equiv$ well known sine series.

Instantaneous velocity of a planet

The *mandaphala* or "equation of centre" correction

- P_0 – mean planet
- P – true planet
- θ_0 – mean longitude
- θ_{MS} – true longitude called the *manda-sphuṭa*.



- The true longitude of the planet is given by

$$\theta = \theta_0 \pm \sin^{-1}\left(\frac{r}{R}\sin M\right)$$

where M (*manda-kendra*) $= \theta_0 -$ longitude of apogee

- The second term in the RHS, known as *manda-phala*, takes care of the eccentricity of the planetary orbit.

Instantaneous velocity of a planet

Derivative of $\sin^{-1}$ function

The instantaneous velocity of the planet called $tātkālikagati$ is given by Nīlakaṇṭha in his $Tantrasaṅgraha$ as follows:

चन्द्रबाहुफलवर्गशोधितत्रिज्यकाकृतिपदेन संहरेत् ।
तत्र कोटिफललिप्तिकाहतां केन्द्रभुक्तिरिह यच्च लभ्यते ॥

If M be the $manda\text{-}kendra$, then the content of the above verse can be expressed as

$$\frac{d}{dt}\left[\sin^{-1}\left(\frac{r}{R}\sin M\right)\right] = \frac{\frac{r}{R}\cos M\,\frac{dM}{dt}}{\sqrt{1-\left(\frac{r}{R}\sin M\right)^2}} \tag{34}$$

Instantaneous velocity of a planet

Derivative of the ratio of two functions

Some of the astronomers in the Indian tradition including $Munjāla$ had proposed the expression for $mandaphala$ to be

$$\Delta\theta = \frac{\frac{r}{R}\sin M}{\left(1 - \frac{r}{R}\cos M\right)}, \tag{35}$$

According to Acyuta, the correction to the mean velocity of a planet to obtain its instantaneous velocity in this case is given by

$$\frac{\left(\frac{r}{R}\cos M\right) + \frac{\left(\frac{r}{R}\sin M\right)^2}{\left(1-\frac{r}{R}\cos M\right)}}{\left(1 - \frac{r}{R}\cos M\right)}\frac{dM}{dt}, \tag{36}$$

which is nothing but the derivative of (**??**).

Concluding Remarks

- It is clear that major discoveries in the foundations of calculus, mathematical analysis, etc., did take place in Kerala School (14-16 century).
- Besides arriving at the infinite series, that the Kerala astronomers could manipulate with them to obtain several forms of rapidly convergent series is indeed remarkable.
- While the procedure by which they arrived at many of these results are evident, there are still certain grey areas (derivative of sine inverse function, ratio of two functions)
- Many of these achievements are attributed to Mādhava, who lived in the 14th century (his works ?).
- Whether some of these results came to be known to the European mathematicians ? ? . . . .

Thanks!

THANK YOU !