Cakravāla

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Cakravāla Method

Let the Vargaprakṛti be $Nx^2 + C = y^2$,
where x is the smaller root (hrasva),
y is the greater root (jyeṣṭha),
and C is the constant (kṣepaka).

Cakravāla Method

hrasva-jyeṣṭha-pada-kṣepān
bhājya-prakṣepa-bhājakān
kṛtvo kalpyo guṇaḥ tatra tathā prakṛti tat cyute
guṇavarge prakṛtyaḥ une athavā alpaṁ śeṣakaṁ yathā

Assume (integer) values for smaller root, greater root and the constant.
Take them to be the dividend, constant and divisor (in order of a kuṭṭaka).
Solve it and find the values of guna and labdhi. Subtract the square of guna from prakṛti or vice versa so that the remainder is the smallest.

Cakravāla Method

tattu kṣepahṛtaṁ kṣepo vyastaḥ prakṛti tat cyute
guṇa labdhiḥ padam hṛsvaṁ tataḥ jyeṣṭaḥ mataḥ asakṛt
tyaktvā pūrvapada kṣepān cakravālaṁ idam jaguḥ
catur dva-ekayutā evaṁ abhinne bhavataḥ pade

Divide this difference by the previous argument(constant) and obtain the new value of the constant.
If the square of the guna is subtracted from the prakṛti, its sign changes.
Assume the guna so obtained as the new value of the smaller root and find the new greater root with prakṛti and constant.
Repeat the process till the constant becomes 4,2 or 1. This is called 'cakravāla'.

caturdvi kṣepa mūlābhyāṁ rūpa kṣepārtha bhāvanā

You get the constant unity from the value of the constant of 4 or 2 or 1 by Bhāvanā.

Procedure

Let the equation to be solved be $Pq^2 + 1 = r^2$

1. Choose s such that $Pq^2 + s = r^2$, so that for $q = 1$, $P + s$ is a perfect square. Get r from $r^2 = Pq^2 + s$.
2. If s=1 or 2 or 4, (q, r) is the solution for $Pq^2 + s = r^2$. Go to step 9.
3. Otherwise, solve the kuṭṭaka $qx + r = sy$.
4. Let $(X, Y) = (x + ms, y + mq)$, m=1,2,3,... be the possible solutions.
5. Choose X such that $P - X^2$ is minimum.
6. Calculate new value of s as $s_1 = (-1).\frac{P-X^2}{s}$.
7. Now find r_1 from $Pq_1^2 + s_1 = r_1^2$, where $q_1 = Y$.
8. Let (q_1, r_1, s_1) be the new values of (q, r, s). Go to step 2.
9. Use Bhāvanā to solve this equation.

Justification

Let the equation to be solved be $Pq^2 + 1 = r^2$
Choose s such that $Pq^2 + s = r^2$, so that for $q = 1$, $P + s$ is a perfect square. Get r from $r^2 = Pq^2 + s$.

- If $s = 1$, we are done.
- If s = -1, use the rule R(1) viz.

  If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 + k = y^2$
  then $Nx^2 + k^2 = y^2$ will have the solution
  $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

  to get the solution of $Pq^2 + 1 = r^2$

Justification ...

- If $s = \pm 2$, use the rule R(1) viz.

  If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 + k = y^2$ then $Nx^2 + k^2 = y^2$ will have the solution $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

  to get the solution of $Pq^2 + 4 = r^2$, and then use the rule R(2) viz.

  If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + c = y^2$ then the solution of $Nx^2 + \frac{c}{k^2} = y^2$ is given by $(x, y) = (\frac{\alpha}{k}, \frac{\beta}{k})$, with $k = 2$

  to get the solution of $Pq^2 + 1 = r^2$

Justification ...

- If $s = 4$, use the rule R(2) above viz.

  If $(x, y) = (\alpha, \beta)$ is a solution for the equation $Nx^2 + c = y^2$ then the solution of $Nx^2 + \frac{c}{k^2} = y^2$ is given by t get the solution of $Pq^2 + 1 = r^2$

- If $s =$ -4, use the rule R(2) above with $k = 2$ to get the solution of $Pq^2 - 1 = r^2$, and then use rule R(1) above viz.

  If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 + k = y^2$ then $Nx^2 + k^2 = y^2$ will have the solution $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

  to get the solution of $Pq^2 + 1 = r^2$

Justification ... Contd

- If s is other than $\pm$ 1,2,4, then we need to find some better approximation of q and s such that s is close to 1.

  Let the solution for $Pq^2 + s = r^2$ be $(q, r) = (1, A)$, where $s = A^2 - P$
  Let another solution for $Pq^2 + k = r^2$ be $(q, r, k) = (x, y, k)$

  Then following Bhāvanā, we have
  $(q, r) = (A.x + y, P.x + A.y)$ as the solution for $Pq^2 + s.k = r^2$
  Or, $(q, r) = (\frac{A.x+y}{s}, \frac{P.x+A.y}{s})$ as the solution for $Pq^2 + \frac{k}{s} = r^2$

  Thus we have two diophantine linear equations viz.
  $q.s = y + A.x$, and
  $r.s = P.x + A.y$.

Justification ... Contd

Let us now consider the first equation:
$q.s = y + A.x - (1)$

Now we solve this diophantine equation, for (A, q), where
$(x, y, s) = (1, a, a^2 - P)$.
$q.(a^2 - P) = a + A$ for the pair of values (A, q).

We will have infinite solutions of the form $(A + m(a^2 - P), q + m)$
for integer values of m.
We choose that pair (A, q) such that $s = A^2 - P$ is the smallest.
For this value of q, now we find r from $Pq^2 + s = r^2$, or

from $r.s = P.x + A.y$, where $(x, y, s) = (1, a, a^2 - P)$.

And new value of s as $s = (-1)\frac{A^2 - P}{s}$.

Example

This is illustrated by the following example.
kā sapta śaṣṭi guṇitā kṛtiḥ eka yuktā
kā ca ekaṣaṣṭi nihatā ca sakhe sarūpā
syāt mūladā yadi kṛti prakṛtiḥ nitāntam
tvat cetasi pravada tāta tatālatāvat

If you are clear and firm in your mind, tell me the following. What is that number, square of which, when multiplied by 67 or 61 and increased by 1 yields a square root?

Find the solutions of $67q^2 + 1 = r^2$ and $61q^2 + 1 = r^2$.

Example1: Solution

The equation to be solved is $67q^2 + 1 = r^2$

1 Choose s such that $67q^2 + s = r^2$, and for $q = 1$, $67 + s$ is a perfect square.
$s = -3$.
$r^2 = 67 - 3$. Hence $r = 8$.

- First Iteration

3.1 Solve the kuṭṭaka $x + 8 = 3y$. (Ignore the sign of s).
4.1 $(X, Y) = (1, 3), (4, 4), (7, 5), (10, 6), ...,$
$(1 + 4m, 3 + m)$, m=0,1,2,3,... are the possible solutions.
5.1 Choose X such that $67 - X^2$ is minimum.
$67 - 7^2$ is minimum.
6.1 New value of s is
$s_1 = (-1).\frac{67-7^2}{-3} = 6$.
7.1 Now from $67.5^2 + 6 = r_1^2$, $r_1^2 = 41^2$.
Hence $r_1 = 41$.
8.1 Let $(q, r, s) = (5, 41, 6)$.

Example1: Solution ...

- Second Iteration
  - 3.1 Solve the kuṭṭaka $5x + 41 = 6y$.
  - 4.1 $(X, Y) = (5, 11), (11, 16), ...,$
    $(5 + 6m, 11 + 5m)$, m=0,1,2,3,... are the possible solutions.
  - 5.1 Choose X such that $67 - X^2$ is minimum.
    $67 - 5^2$ is minimum.
  - 6.1 New value of s is
    $s_1 = (-1).\frac{67-5^2}{6} = -7$.
  - 7.1 Now from $67 * 11^2 - 7 = r_1^2$, $r_1^2 = 90^2$.
    Hence $r_1 = 90$.
  - 8.1 $(q, r, s) = (11, 90, 7)$.

Example1: Solution ...

- Third Iteration
  - 3.1 Solve the kuṭṭaka $11x + 90 = 7y$.
  - 4.1 $(X, Y) = (2, 16), (9, 27), ...,$
    $(2 + 7m, 9 + 11m)$, m=0,1,2,3,... are the possible solutions.
  - 5.1 Choose X such that $67 - X^2$ is minimum.
    $67 - 9^2$ is minimum.
  - 6.1 New value of s is
    $s_1 = (-1).\frac{67-9^2}{-7} = -2.$
  - 7.1 Now from $67 * 27^2 - 2 = r_1^2$, $r_1^2 = 221^2$.
    Hence $r_1 = 221$.
  - 8.1 $(q, r, s) = (27, 221, 2)$.

Example1: Solution ...

s=2, and hence
$(27, 221)$ is the solution for $67q^2 - 2 = r^2$.

Following the rule in Bhāvanā viz.
If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 + k = y^2$
then $Nx^2 + k^2 = y^2$ will have the solution
$(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

$67q^2 + 4 = r^2$ will have a solution given by
$(2 * 27 * 221, 67 * 27^2 + 221^2)$, or
$(11934, 97684)$.

Following the rule in Bhāvanā viz.
If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 + k^2 = y^2$
then $Nx^2 + 1 = y^2$ will have the solution
$(x, y) = (\frac{\alpha}{k}, N\frac{\alpha^2}{k})$

$67q^2 + 1 = r^2$ will have a solution given by $(\frac{11934}{2}, \frac{97684}{2}) =$
$(5967, 48842)$.

Example2: Solution

The equation to be solved is $61q^2 + 1 = r^2$

1 Choose s such that $61q^2 + s = r^2$, ans for $q = 1$, $61 + s$ is a perfect square.
$s = 3$.
$r^2 = 61 + 3$. Hence $r = 8$.

- First Iteration

3.1 Solve the kuṭṭaka $x + 8 = 3y$.
4.1 $(X, Y) = (1, 3), (4, 4), (7, 5), (10, 6), ...,$
$(1 + 4m, 3 + m)$, m=0,1,2,3,... are the possible solutions.
5.1 Choose X such that $61 - X^2$ is minimum.
$61 - 7^2$ is minimum.
6.1 New value of s is
$s_1 = (-1).\frac{61-7^2}{3} = -4$.
7.1 Now from $61 * 5^2 - 4 = r_1^2$, $r_1^2 = 39^2$.
Hence $r_1 = 39$.
8.1 Let $(q, r, s) = (5, 39, 4)$.

Solution ... Contd

s=4, and
$(5, 39)$ is the solution for $61q^2 - 4 = r^2$.

Following the rule in Bhāvanā viz.
If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 - k^2 = y^2$
then $Nx^2 - 1 = y^2$ will have the solution
$(x, y) = (\frac{\alpha}{k}, \frac{\beta}{k})$

$61q^2 - 1 = r^2$ will have a solution given by $(\frac{5}{2}, \frac{39}{2})$

Following the rule in Bhāvanā viz.
If $(x, y) = (\alpha, \beta)$ is a solution of $Nx^2 + k = y^2$
then $Nx^2 + k^2 = y^2$ will have the solution
$(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$

$61q^2 + 1 = r^2$ will have a solution given by $(2 * \frac{5}{2} * \frac{39}{2}, 61 * \frac{5}{2}^2 + \frac{39}{2}^2)$
$= (\frac{195}{2}, \frac{1523}{2})$.

Solution ... Contd

Following Bhāvanā again,
$(q, r) = (2 * \frac{195}{2} * \frac{1523}{2}, 61 * \frac{195}{2}^2 + \frac{1523}{2}^2)$
$= (\frac{296985}{2}, \frac{2319527}{2})$.

Following Bhāvanā once again,
$(q, r) = (226153980, 11766319049)$.

More rules

rūpa śuddhau khiloddiṣṭaṁ
varga yoge guṇe na cet

If the argument is negative, the equation can not be solved unless the dividend is the sum of two squares.
i.e., for $Nx^2 - 1 = y^2$ has no integer solution unless $N = a^2 + b^2$

akhile kṛti mulābhyām
dvidhā rūpaṁ vibhājitaṁ
dvidhā hrasva padaṁ jyeṣṭaṁ
tato rūpa viśodhane
pūrvavad vā prasādhyete
pade rūpa viśodhane

In the case of argument being -1, the roots of the sqaures whose sum is the prakṛti, should be obtained. There will be two such values. The reciprocals of these values will be the two smaller roots. From these smaller vaues, the larger values can be estimated. In other words, for the equation $Nx^2 - 1 = y^2$, and

$N = a^2 + b^2$
the two smaller roots are $x = \frac{1}{a}$ and $x = \frac{a}{b}$.
The larger roots can then be obtained from the given equation.

Example

trayodaśa guṇo vargo nirekaḥ kaḥ kṛtir bhavet
ko vāṣṭa guṇito vargo nnireko mūladā vada

Which is the number, whose square multiplied by 13 and reduced by one becomes a square?
Also tell me the number whose square multipled by 8 and reduced by one gives a suqare root.

Solution

There are two equations given
$13x^2 - 1 = y^2$, and
$8x^2 - 1 = y^2$
Example 1: $13x^2 - 1 = y^2$
The argument is -1. The prakṛti is 13.
Since $13 = 3^2 + 2^2$
we have $x = \frac{1}{3}$, and $x = \frac{1}{2}$, as two possible solutions.
Substituting these values, we get $y = \frac{2}{3}$ and $y = \frac{3}{2}$.
With $x = \frac{1}{2}$ and $y = \frac{2}{3}$, using bhāvanā, the roots of $13x^2 + 1 = y^2$ are
$x_1 = 2 * \frac{1}{2} * \frac{3}{2} = \frac{3}{2}$
$y_1 = 13 * \frac{1}{4} + \frac{9}{4} = \frac{11}{2}$

Using bhāvanā again, with the solutions $(x, y) = (\frac{1}{2}, \frac{3}{2})$ and $(x, y) = (\frac{3}{2}, \frac{11}{2})$ for $13x^2 - 1 = y^2$ and $13x^2 + 1 = y^2$, we get,
$x_1 = \frac{1}{2} * \frac{11}{2} + \frac{3}{2} * \frac{3}{2} = \frac{20}{4} = 5$
$y_1 = 13 * \frac{3}{4} + \frac{33}{4} = \frac{72}{4} = 18$

Solution

Example 1: $8x^2 - 1 = y^2$
The argument is -1. The prakṛti is 8.
Since $8 = 2^2 + 2^2$
we have $x = \frac{1}{2}$ for both possible solutions.
Substituting this value in the given equation, we get $y = 1$.
So the possible solution is $(\frac{1}{2}, 1)$.

Now, With $x = \frac{1}{2}$ and $y = 1$, using bhāvanā, we get the solutions for $13x^2 + 1 = y^2$.
$x_1 = 2 * \frac{1}{2} * 1 = 1$
$y_1 = 8 * \frac{1}{4} + 1 = 3$

Now, With $x = \frac{1}{2}$ and $y = 1$, for $8x^2 - 1 = y^2$ and $(1, 3)$ for $8x^2 + 1 = y^2$, using bhāvanā, we get the solutions for $13x^2 - 1 = y^2$ as.
$x_2 = \frac{1}{2} * 3 + 1 * 1 = \frac{5}{2}$
$y_2 = 8 * \frac{1}{2} * 1 + 1 * 3 = 7$
So we do not get an integer solution.

example

ko vargaḥ śadguṇaḥ tryādhyo
dvādaśādhyo athavā kṛtiḥ
yuto vā pañca saptatyā
triśatyā vā kṛtir bhavet

Which is that number whose square multiplied by 6 and increased by 3, 12, 75 and 300 respectively becomes square numbers.

$6x^2 + 3 = y^2$; Solution $(1, 3)$
$6x^2 + 12 = y^2$; Solution $(2, 6)$
$6x^2 + 75 = y^2$; Solution $(5, 15)$
$6x^2 + 300 = y^2$; Solution $(10, 30)$

Rule

varga cchinne guṇe hrasvam tat padena vibhājayet.

If the guṇa (prakṛti) is divided by the square of any number, then the smaller root of the reduced equation should be divided by the square root of the square used for division, to get the correct root of the original euqation.

For example, in the equation $8x^2 + 1 = y^2$, the prakṛti 8 can be divided by 2^2 to simplify the given equation as $2x^2 + 1 = y^2$. Root of this reduced euqation are $(2, 3)$.

So the roots of the original equation are obtained by dividing the smaller root by the square root of 4, i.e., $(\frac{2}{2}, 3) = (1, 3)$.

dvātriṁśat guṇito vargaḥ kaḥ sa-eko mūlado vada

Tell the number, the square of which when multiplied by 32 and increased by 1 gives a perfect square. The equation is

$32x^2 + 1 = y^2$.
Dividing the prakṙti by 16, we get $2x^2 + 1 = y^2$.

This has a solution $(2, 3)$.
Hence the solution of the original equation is $(\frac{2}{4}, 3)$.
Using bhāvanā, we get
$(2 * \frac{2}{4} * 3, 32 * \frac{2}{4}\frac{2}{4} * 3 * 3) = (3, 17)$.

Rule

iṣṭa bhakto dvidhā kṣepa iṣṭonāḍhyo dalīkṛtāḥ
guṇa mūla hṛtaḥ cādyo hṛasva jyeṣṭe kramālpade

If the prakṛti(guṇa) is a square number, divide the constant argument by any optional number and write the quotient at 2 places. From one subtract the optional number and add it to the second. Divide both by 2. Dividing the first one by the square root of the guṇa gives the smaller root x and the second one gives the larger root.
For example, consider $16x^2 + 10 = y^2$. The guṇa is a perfect square.
Divide the constant say by 2. We get $\frac{10}{2} = 5$. Subtracting and adding 2 from 5 and then dividing it by 2, we get $\frac{5-2}{2}$ and $\frac{5+2}{2}$. Dividing the first one by square root of 16, we get $\frac{3}{2*4}$ as value of x, and $y = \frac{7}{2}$.