Math for Poets and Drummers

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Around the sixth century BC, the ancient Greeks discovered a seemingly mystical correspondence between musical intervals with a pleasing sound and ratios of whole numbers: the principal tones of a musical scale are produced by fretting a string at points that divide its length into simple ratios. Although it would take more than two millenia for this phenomenon to be fully explained, an association between musical harmony and proportions was firmly established in Western thought. The ancients believed musical harmony to be a compelling demonstration of mathematical order in the universe—indeed, through medieval times, the study of proportions was closely linked to the study of music.

The fact that scholars in ancient India made significant mathematical discoveries while analyzing the rhythms, or *meters*, of poetry has been largely overlooked in the West. In English, a meter is a pattern of stressed and unstressed syllables. For example, Shakespeare's plays are written in a meter called *iambic pentameter*, five pairs of alternating unstressed and stressed syllables to a line. In Sanskrit, the classical language of India, a meter is a pattern of short and long syllables that dictates the rhythm of a poem. While there are only about a dozen English meters, there are hundreds of meters in Sanskrit. Many meters are associated with specific religious rituals.

What would a catalog of all possible meters look like? Of course, it would be infinitely large; the challenge is to come up with a finite set of instructions—an *algorithm*—that generates all viable patterns. How can we check that every meter belongs somewhere in this catalog? Is there a shorthand we can use to remember a metrical pattern? Indian scholars answered these questions in ingenious ways.

The search for ways to list and classify meters led to important mathematical discoveries: Pascal's triangle, the Fibonacci numbers, and even the rudiments of the binary number system. The discoveries of these structures in India predated those in the West, sometimes by several centuries. They used recursion and iteration—essential computer programming techniques—to generate lists of rhythms. The fact that several authors solved the same problems in different ways reveals the depth of Indian mathematical development at the time. Their discoveries also apply to rhythm patterns in traditional and popular music.

The binary representation of meter

Several features of Sanskrit poetry are particularly mathematical. Sanskrit meters are patterns of long (*guru*) and short (*laghu*) syllables, with a long syllable having twice the length of a short one (in English, syllables have no numerical value). Sanskrit meters fall into two categories: meters in which the number of syllables in a line are fixed, and meters in which the duration of a line is fixed, but not the number of syllables. This distinction naturally raises several mathematical questions.

Pingala is credited with the first work on prosody, the systematic study of meter. We know very little about him. Some modern scholars think he lived around 500 BC and was the nephew of the great Sanskrit grammarian Panini; others claim he lived around 200 BC. The earliest definitive reference to his writing comes in the third century AD [11]. Moreover, it is not clear whether the works attributed to him were written by the same person, or whether, as in the case of Pythagoras, some were written by his followers. Pingala's writings took the form of short, cryptic verses, or *sūtras*, which served as memory aids for a larger set of concepts passed on orally.

We rely on medieval commentators for much of the interpretation of Pingala's work. They include Halāyudha (13th century) and Kedāra Bhatt (8th century). Bhatt solves the same problems Pingala does, but uses such different methods that modern scholars disagree on whether or not his works are commentaries on Pingala [11, 5].

Pingala studied meters with a fixed number of syllables. It is easy to discover by experimentation that there are two meters of one syllable, four meters of two syllables, and eight meters of three syllables (Western prosodists stopped here). Listing meters for any number of syllables is more of a challenge.

Pingala's sūtras address four problems:

> **Problem 1.** How can we systematically list all the patterns of n syllables for any n?
> **Problem 2.** Suppose a pattern is erased from this list. How can we recover the missing pattern?
> **Problem 3.** Given any pattern, how can we find its position on the list without recreating the entire list?
> **Problem 4.** What is the total number of patterns of n syllables?

Problem 1: listing the patterns of n syllables.

Pingala gave instructions on how to list the patterns of n syllables in a table he called *prastāra*, or expansion. Pingala's first *sūtra* states that the expansion of one syllable has two elements (a long syllable and a short syllable, represented by the letters ऽ and I, respectively). His second *sūtra* observes that the expansion of two syllables is the one-syllable expansion "mixed with itself." That is, "mix" ऽ and I with ऽ to get ऽऽ and Iऽ;

1	ऽ	1	ऽऽ	1	ऽऽऽ	1	ऽऽऽऽ	9	ऽऽऽ।
2	।	2	।ऽ	2	।ऽऽ	2	।ऽऽऽ	10	।ऽऽ।
		3	ऽ।	3	ऽ।ऽ	3	ऽ।ऽऽ	11	ऽ।ऽ।
		4	।।	4	।।ऽ	4	।।ऽऽ	12	।।ऽ।
				5	ऽऽ।	5	ऽऽ।ऽ	13	ऽऽ।।
				6	।ऽ।	6	।ऽ।ऽ	14	।ऽ।।
				7	ऽ।।	7	ऽ।।ऽ	15	ऽ।।।
				8	।।।	8	।।।ऽ	16	।।।।

Figure 1: Expansions of one-, two-, three-, and four-syllable meters, with indices

mix ऽ and । with । to get ऽ। and ।।. To get the three-syllable expansion, append ऽ to the end of the two-syllable expansion, then do the same for ।. The final *sūtra* states that there are eight patterns of three syllables. Presumably, we are to again combine the three-syllable patterns separately with ऽ and । to get the expansion of four-syllable patterns. These instructions generalize to any length of pattern. Prastāras of one through four syllables are shown in Figure 1.

Kedāra Bhatt gives an completely different algorithm that nonetheless generates the list of n-syllable patterns in the same order Pingala uses [11, 5]. The first pattern on the list consists of n long syllables. Suppose you are given any pattern on the list (for example, ।।ऽ।ऽऽऽऽ). To get the next pattern, start from the left by writing long syllables:

When you reach the position of the first long syllable in the previous pattern, write a short syllable:

।	।	ऽ	।	ऽ	ऽ	ऽ	ऽ
ऽ	ऽ	।	-	-	-	-	-

Then recopy the rest of the previous pattern:

।	।	ऽ	।	ऽ	ऽ	ऽ	ऽ
ऽ	ऽ	।	।	ऽ	ऽ	ऽ	ऽ

The list ends with the pattern of all short syllables (without this stipulation, the list repeats in an infinite loop, since applying the algorithm to the pattern of n short syllables produces the pattern of n long syllables).

Pingala's algorithm follows naturally from the observation that the list of $(n-1)$-syllable patterns is nested (twice) within the list of n-syllable patterns. Bhatt's algorithm is less obvious; it is perhaps easiest to derive from the routine described in his solution to Problem 3 (discussed below).

Combinatorial sequence generation is the process of systematically listing structures with a given property. Computer-science guru Donald E. Knuth credits Sanskrit prosodists with "the first-ever explicit algorithm for combinatorial sequence generation" [4]. A significant difference between the two writers is that Bhatt's algorithm is *iterative*—that is, it gives instructions to get from one pattern on the list to the next—while Pingala's algorithm is *recursive*—it generates the entire list of n-syllable patterns from the list of $(n-1)$-syllable patterns, so that the list of patterns of any given length may be generated by repeatedly invoking the same routine. Both iteration and recursion are fundamental computer programming techniques. A fascination with recursion appears in Indian art and religion from ancient times. For example, the medieval Kandariya Mahadeva temple (figure 2) contains several miniature copies of itself.

Problem 2: recovering a lost pattern.

Suppose a pattern is erased from the list. How can we recover it without having to regenerate the entire list? Bhatt does not address this problem, as his algorithm generates the missing row from the previous pattern. However, this is a serious problem for Pingala.

Pingala's pattern recovery algorithm assumes that we know the position of the missing pattern, which we will call its *index* (see Figure 1). He gives the following instructions: if the index can be halved, halve it and write ऽ; otherwise, write ।, add one, and halve the result. Repeat the process, writing from left to right, until the pattern has the correct number of syllables.

To understand why Pingala's algorithm works, let w represent a string of n characters drawn from the set {ऽ,।} and let $\operatorname{ind} w$ denote its index. Observe that the index of w is odd if w starts with ऽ and even if w starts with ।. If we remove the first syllable of pattern w to get a new pattern, w', then

$$\operatorname{ind} w' = \begin{cases} (\operatorname{ind} w + 1)/2 & \text{if } \operatorname{ind} w \text{ is odd} \\ (\operatorname{ind} w)/2 & \text{if } \operatorname{ind} w \text{ is even} \end{cases}$$

Suppose we know the index of pattern p (but not the pattern itself), and wish to recover the pattern. Since we use a repeated routine, it is convenient to rename p as p_n. We recover the syllables of p_n one at a time; at each point the string of unknown syllables is one shorter, and we call these successively shorter strings $p_{n-1}, p_{n-2}, \ldots, p_1$. If $\operatorname{ind} p_n$ is odd, p_n begins with ऽ; therefore, $p_n =$ ऽp_{n-1}, where $\operatorname{ind} p_{n-1} = (\operatorname{ind} p_n + 1)/2$. If $\operatorname{ind} p_n$ is

Figure 2: Recursion in Indian architecture: the Kandariya Mahadeva temple

even, p_n begins with ।; therefore, $p_n =$ ।p_{n-1}, where $\operatorname{ind} p_{n-1} = \operatorname{ind} p_n/2$. We now know the index of p_{n-1} in the list of patterns of length $n-1$. Repeat the algorithm until all the characters of p_n have been generated.

The following steps show that the fifth pattern of five syllables is ऽऽ।ऽऽ:

pattern	ind p_i	parity
p_5	5	odd
ऽp_4	$(5+1)/2 = 3$	odd
ऽऽp_3	$(3+1)/2 = 2$	even
ऽऽ।p_2	$2/2 = 1$	odd
ऽऽ।ऽp_1	$(1+1)/2 = 1$	odd
ऽऽ।ऽऽ		

Problem 3: finding the index of a pattern.

Now, suppose you are given a pattern. Where does it belong on the list? Pingala's indexing process reverses the algorithm he developed for Problem 2. The index of the pattern of all long syllables is one. For any other pattern, start with the first short syllable from the right. The instruction is simply "multiply by two" (in order for the algorithm to work, the starting number must be one). If the next syllable on the left is ऽ, again multiply the resulting number by two; otherwise, multiply it by two and subtract one. Repeat this process until the leftmost character is reached.

This procedure stems from the observation that if w is a pattern,

ind ऽw = $2 \operatorname{ind} w - 1$

ind ।w = $2 \operatorname{ind} w$

Since adding any number of ऽ's to the end of a word does not change its index, Pingala's process begins with the first short syllable from the right. While the original pattern is recreated by adding one syllable at a time on the left, the algorithm keeps track of the index of the current pattern.

The following steps show that the index of ऽऽ।ऽऽ is five.

pattern	index
ऽऽ	1
।ऽऽ	$2 \cdot 1 = 2$
ऽ।ऽऽ	$2 \cdot 2 - 1 = 3$
ऽऽ।ऽऽ	$2 \cdot 3 - 1 = 5$

Bhatt's algorithm for finding the index of a given pattern is again strikingly different from Pingala's. It stems from the observation that if w is a meter of n syllables,

$$\begin{aligned} \operatorname{ind} w\text{ऽ} &= \operatorname{ind} w \\ \operatorname{ind} w\text{।} &= \operatorname{ind} w + 2^n. \end{aligned}$$

Accordingly, Bhatt assigned a place value to each syllable. Reading from the left, the first place has value one, the second has value two, the third has value four, and so on, so that the value of the ith place is 2^{i-1}. Bhatt observed that the index of a pattern is one more than the sum of the place values of its short syllables.

For example, the index of ।ऽ।ऽ is six, because short syllables fall in the first and third columns:

$$\begin{array}{ccccccccccc} 1 & + & 1 & + & 0 \cdot 2 & + & 4 & + & 0 \cdot 8 & = & 6 \\ & & \text{।} & & \text{ऽ} & & \text{।} & & \text{ऽ} & & \end{array}$$

Perhaps Bhatt was predisposed to use a positional indexing system, as the positional decimal numbers are thought to have been adopted in India close to the century of his birth [3].

We may now return to Bhatt's somewhat opaque solution to Problem 1. Suppose we know the kth pattern of n syllables, and wish to write the $(k+1)$st pattern. We define $k_1, k_2, \ldots, k_n$ by

$$k_i = \begin{cases} 0 & \text{if the } i\text{th syllable is ऽ} \\ 1 & \text{if the } i\text{th syllable is ।} \end{cases}$$

Then the index of the kth pattern is

$$1 + k_1 + 2k_2 + 4k_3 + \ldots + 2^{n-1}k_n,$$

If $k_1, k_2, \ldots, k_n$ begins with a string of 1s, to get the $(k+1)$st pattern, change all of these to 0s, replace the first 0 with 1, and leave the rest of the sequence alone.

Problem 4: counting patterns of n syllables.

The fourth problem both Pingala and Bhatt tackled involves counting the possible poetic meters of a fixed number of syllables [1]. In other words, they wrote procedures to evaluate 2^n. Bhatt gives two algorithms [5]. The first is based on his solution to the index-finding problem. He observes that the index of the pattern of n short syllables is 2^n. Using his algorithm, this equals one plus the sum of the positional value of each syllable; in other words,

$$2^n = 1 + \sum_{i=1}^{n} 2^{i-1}.$$

Bhatt's second algorithm involves summing the binomial coefficients ${}_nC_r$; we will discuss these in the next section.

Pingala, on the other hand, again gives a recursive algorithm based on the observation that

$$2^n = \begin{cases} (2^{n/2})^2 & \text{(used if } n \text{ even)} \\ 2^{n-1} \cdot 2 & \text{(used if } n \text{ odd)} \end{cases}$$

(of course, both statements are equivalent, but Pingala would not have been able to evaluate $2^{n/2}$ for n odd) [5]. For example, we calculate 2^9:

$$2^9 = 2^8 \cdot 2 = (2^4)^2 \cdot 2 = ((2^2)^2)^2 \cdot 2.$$

The binary number system

In some ways, Pingala and Bhatt anticipated the development of the binary number system, which was not fully described until Gottfried Leibniz did so in the seventeenth century.[1] The normal decimal-to-binary conversion procedure is quite similar to Pingala's process for finding an unknown pattern given its index. In this case, the decimal number serves as the "index." Let b be a string of ones and zeros, and let $\operatorname{dec} b$ be the decimal value of the binary number b represents. We may concatenate b with either 0 or 1; note that $\operatorname{dec} b0 = 2 \operatorname{dec} b$ and $\operatorname{dec} b1 = 2 \operatorname{dec} b + 1$. Therefore, to find b if its decimal value is known, divide the decimal by two, write the remainder, and continue this process, writing the successive remainders on the left.

[1] The binary number system is a base-two positional number system. It has two digits, 0 and 1, and its place values are powers of two. Thus, the decimal numbers 1, 2, 8, and 11 have binary representation 1, 10, 1000, and 1011, respectively.

In contrast, Bhatt's algorithm, which assigns a positional value to each syllable, recalls the binary-to-decimal conversion formula

$$\operatorname{dec} b_n b_{n-1} \ldots b_1 b_0 = \sum_{i=1}^{n} b_i 2^i.$$

Although these relationships are intriguing, there are substantial differences between Pingala and Bhatt's indexing system and the binary numbers. There is no evidence that either author considered his indexing procedure to be a number system; it was not used to perform computations, or indeed to count anything other than poetic meters. The convention of assigning the index one, rather than zero, to the first pattern makes computations problematic (giving ऽऽ...ऽऽthe index zero was not an option at the time—in fact, we have no record that the Indians considered zero a number before the fifth century AD). The correspondence between metrical patterns and their indices is one-to-one only if the number of syllables is fixed (this is like considering 1, 01, and 001 to be distinct numbers). The other dissimilarity—which arises from the ordering of poetic meters—is that the highest-valued columns are on the right. The positional decimal number system developed in India places the highest-valued columns on the left.

Pingala developed a completely different way of cataloging meters that is much more common—in fact, it is used by poets and drummers today. This is considered in the last section of this chapter.

Pascal's Triangle and the Expanding Mountain of Jewels

Pingala is also credited with the discovery of "Pascal's" Triangle in India, which he called the *meruprastāra*, or "the expanding mountain of jewels" (*meru* is a mythical mountain made of gold and precious stones, and *prastāra* is the word for expansion). However, precisely which problem in prosody led him to this discovery is uncertain. Some medieval commentators interpret the numbers in the meruprastāra as being the number of combinations of n syllables, taken one at a time, taken two at a time, and so on (each syllable is considered different, rather than just long or short). When each list counting combinations of r syllables drawn from sets of n syllables is arranged horizontally, and successive lists are stacked, the numbers form a triangular array that one can extend indefinitely:

$$\begin{array}{ccccccccc}
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Pingala recognized that each interior number is the sum of the two numbers above it. This array is known as Pascal's triangle—though, of course, it wasn't yet named for Pascal (who was born in France in 1623).

The roof of the Kandariya Mahadeva temple (figure 2) depicts Mount Meru surrounded by smaller copies of itself. Note that the meruprastāra is also made up of smaller copies of itself!

$$\begin{array}{ccccccccc}
 & & & 1 & & - & & & \\
 & & 1 & & 1 & & - & & \\
 & 1 & & 2 & & 1 & & - & \\
1 & & 3 & & 3 & & 1 & & -
\end{array}
\quad + \quad
\begin{array}{ccccccccc}
 & & & - & & 1 & & & \\
 & & - & & 1 & & 1 & & \\
 & - & & 1 & & 2 & & 1 & \\
- & & 1 & & 3 & & 3 & & 1
\end{array}$$

Bhatt discovered the same triangle, but in a different context: he found the number of meters of n syllables having r short syllables. The fact that the two problems produce the same triangle is, of course, no coincidence. The number of ways of choosing r syllables from a collection of n different syllables (${}_nC_r$ in modern notation) equals the number of ways of choosing the r locations for the short syllables within a meter of n short or long syllables. For example, the choice of $\{2, 5, 7\}$ from the collection $\{1, 2, \ldots, 8\}$ corresponds to the eight-syllable meter that has short syllables in positions 2, 5, and 7 (ऽIऽऽIऽIऽ).

Returning to Problem 4, Bhatt's interpretation of the triangle shows that the sum of the entries in the nth row of the triangle gives the total number of meters of n syllables. In modern notation, Bhatt's second solution to Problem 4 is the formula

$$2^n = \sum_{r=0}^{n} {}_nC_r.$$

None of the Indian authors explain the relationship between the addition rule for obtaining successive rows of the meruprastāra and the structure of the meters they represent. However, given Pingala's fondness for recursive rules, he may have observed a one-to-one correspondence between the ways of choosing r objects from a collection of n objects and the ways of choosing either $r - 1$ objects out of $n - 1$ objects or r objects out of $n - 1$ objects. For example, there are ten three-element combinations of $\{1, 2, 3, 4, 5\}$. Partition these into combinations that contain 5 and combinations that do not. If a combination is to contain the element 5, you choose the other two elements from the set {1, 2, 3, 4} in ${}_4C_2$ ways; add to this the ${}_4C_3$ three-element combinations that do not contain 5 to arrive at the equation ${}_5C_3 = {}_4C_2 + {}_4C_3$. We can make a similar argument for meters; in this case, partition the five-syllable meters into those with the fifth syllable short and those with the fifth syllable long.

The 12th-century writer Bhaskara gives yet another algorithm in his *Lilavati* [2]. To find the nth row in the meruprastāra, start by writing the numbers 1, 2, . . ., n, and above them write the numbers n, $n-1$, . . ., 2, 1, like so (shown for $n = 5$):

$$\begin{matrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{matrix}$$

The first number in the row is 1 (this is true for every n). Obtain the other numbers in the row by succesively multiplying and dividing by the numbers you have written:

$$1 \cdot 5/1 = 5; \quad 5 \cdot 4/2 = 10; \quad 10 \cdot 3/3 = 10; \quad 10 \cdot 2/4 = 5; \quad 5 \cdot 1/5 = 1.$$

This algorithm is iterative; you do not have to generate any previous rows in order to find row n.

Although he does not make a connection to the recursive addition rule found by Pingala, Bhaskara comments that the $n$the row of the meruprastāra counts both the number of ways of choosing r of n different objects and the number of ways of arranging r objects of one kind and $n - r$ of another. He also notes that prosody is only one of the possible applications of the meruprastāra.

The Hemachandra-Fibonacci numbers

The 12th-century writer Ācārya Hemachandra also studied poetic meter [9]. Instead of counting meters with a fixed number of syllables, Hemachandra counted meters having a fixed duration, counting short syllables as one beat and long syllables as two beats, as shown in figure 3. The numbers of patterns form the sequence $1, 2, 3, 5, 8, \ldots$. Hemachandra discovered that each entry is found by adding the two previous. In other words, he found the "Fibonacci" numbers—half a century before Fibonacci! Indian poets and drummers know these numbers as "Hemachandra numbers."

Hemachandra explained why the sequence counts the succesive numbers of patterns of length n. Each pattern ending with a short syllable is a pattern of duration $n - 1$ followed

1 beat	2 beats	3 beats	4 beats	5 beats	
I	S	IS	SS	ISS	SSI
	II	SI	IIS	SIS	IISI
		III	ISI	IIIS	ISII
			SII		SIII
			IIII		IIIII

Figure 3: Meters listed by duration

by a short syllable; the number of strings of duration $n-1$ is H_{n-1}, and therefore the number of patterns of duration n that end with a short syllable is H_{n-1}. Each pattern ending with a long syllable begins with a string of syllables of total duration $n-2$, followed by a long syllable. Therefore, there are H_{n-2} of these. Finally, H_n, the total number of patterns of duration n, equals $H_{n-1} + H_{n-2}$. Since $H_1 = 1$ and $H_2 = 2$, we obtain the Hemachandra sequence. This derivation of the Hemachandra-Fibonacci numbers is identical to the "domino-square problem": in how many ways can you tile a $1 \times n$ rectangle with 1×2 dominoes and 1×1 squares? Figure 4 shows a visual solution to this problem.

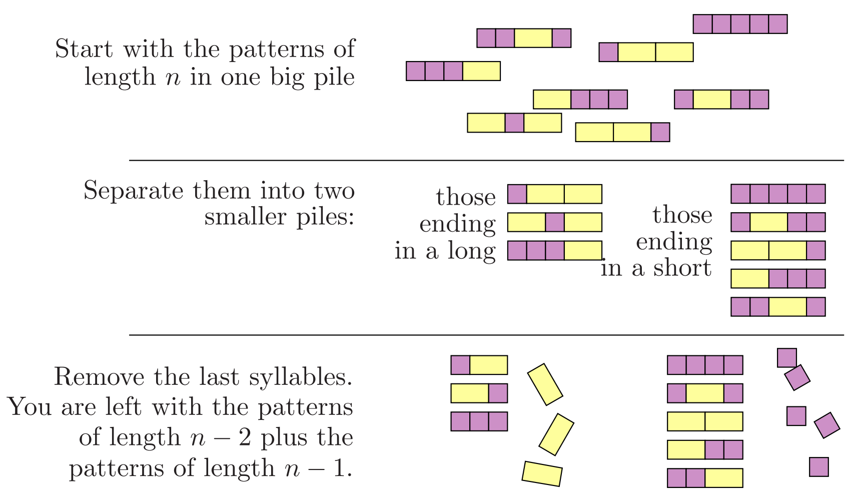

Figure 4: Visual solution to the domino/square problem

It is possible that Pingala was aware of this sequence, as well. The tenth-century commentator Yādava interprets Pingala's rule "and the two mixed" to mean that the patterns of duration n are built up of shorter patterns (in this case, patterns of duration $n-1$ and $n-2$).

The *Prākrta Paiṅgala*, dating from the the early 14th century AD, makes an explicit

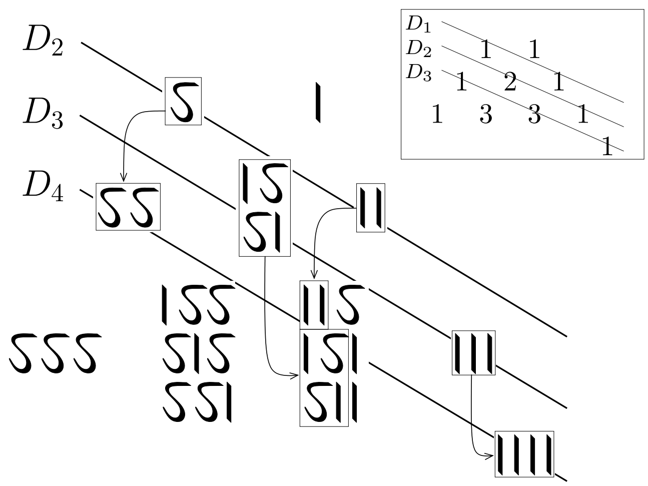

Figure 5: The Hemachandra numbers are sums along diagonals in the meruprastāra

connection between the Hemachandra numbers and Pingala's meruprastāra. The author demonstrated that the Hemachandra numbers are, in fact, sums of numbers on certain *diagonals* in the meruprastāra. Figure 5 provides a visual explanation of this property: the five four-beat rhythms lie on the diagonal labeled D_4. The process of combining rhythms to form entries in the triangle also ensures that the patterns on D_4 are formed from the patterns on D_3 and D_2.

Fibonacci's thirteenth-century "discovery" of the sequence that bears his name about fifty years after Hemachandra's discovery of the same sequence was probably no coincidence. Fibonacci, who was educated in North Africa, was quite familiar with Eastern mathematics (in particular, he introduced the Indian positional number system to the West) [9]. He may have first encountered the "Fibonacci" sequence in the East. However, his explanation of the sequence as the sizes of successive generations of rabbits is not found in India.

Musical rhythm patterns and the Padovan numbers

The poetic meters that Pingala and Hemachandra studied have an analogue in music, and, indeed, many of the rhythms used in classical Indian music have been deeply influenced by the meters of Sanskrit poetry.

In music, rhythm patterns are formed by grouping beats into notes, which play the role of syllables in poetry. A drum is hit on the first beat of each note and silent on the following beats; the length of a note is the number of beats between successive hits. Some types of music, especially dance music, are identified with specific rhythm patterns. Many of these patterns are formed of notes of two or three different durations. For example, in salsa music, which has origins in Cuba, one can hear the pattern called the *clave*. The 12-beat clave rhythm and many other characteristic rhythm patterns from around the world are composed of notes of lengths two or three beats; one also finds many patterns consisting of notes of lengths one or two beats. Figure 6 shows a few examples. To hear some of these patterns, I suggest SongTrellis [6], where you can hear the merengue and cumbia bell parts both separately and in context. The guajira may be familiar as the rhythm of Leonard Bernstein's "America," from West Side Story. I encourage the reader to experiment with creating and listening to rhythm patterns; a good place to start is the web applet Jas's MIDI Hand Drum Rhythm Generator [8].

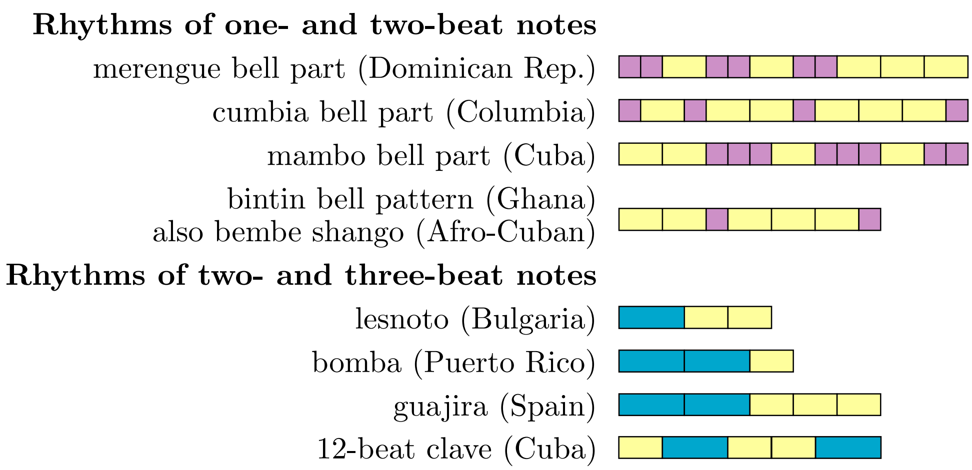

Figure 6: Dance rhythms

Hemachandra discovered the sequence that counts patterns of one- and two-beat notes. What sequence counts patterns consisting of two- and three-beat notes? Here are the first twelve entries of this sequence:

length (n)	1	2	3	4	5	6	7	8	9	10	11	12
number of patterns (P_n)	0	1	1	1	2	2	3	4	5	7	9	12

If P_n is the number of such patterns, then $P_n = P_{n-2} + P_{n-3}$. The proof of this statement is similar to the argument for notes of length one and two. In this case, break the patterns of length n into patterns of length $n-2$ followed by a two-beat note and patterns of length $n-3$ followed by a three-beat note.

Though not nearly as famous as the Hemachandra-Fibonacci numbers, this sequence, named the Padovan numbers, has some interesting properties. It is well known that the limit of the ratios of successive Fibonacci numbers is the golden number $\phi = 1.618\ldots$; the limit of the ratios of successive Padovan numbers is the so-called "plastic number." To find this number, observe that

$$\frac{P_n}{P_{n-1}} = \frac{P_{n-2}}{P_{n-1}} + \frac{P_{n-3}}{P_{n-1}} = \frac{P_{n-2}}{P_{n-1}} + \frac{P_{n-3}}{P_{n-3} + P_{n-4}}.$$

Take the limit as $n \to \infty$ of both sides and let $p = \lim_{n\to\infty} P_n/P_{n-1}$. Then

$$p = \frac{1}{p} + \frac{1}{1 + 1/p}$$

so p is a solution to the cubic equation $p^3 - p - 1 = 0$. The only real root of this equation is the irrational number $p = 1.324\ldots$ (the plastic number). There are several interesting applications of the Padovan numbers—for example, they are related to a spiral of equilateral triangles in the way the Hemachandra-Fibonacci numbers are related to a spiral of squares (see [10] for more applications).

Naming rhythms

Since there are hundreds of Sanskrit meters, remembering the pattern for any particular meter requires some effort. Although Pingala's indexing procedure is mathematically impressive, being able to identify the pattern SSSISI as "number forty-one in the catalog of six-beat rhythms" is not of much practical use.

Pingala's best and most well-known solution to this problem involves the following mapping of groups of three syllables to letters:

m	SSS	*r*	SIS	*t*	SSI	*bh*	SII
y	ISS	*s*	IIS	*j*	ISI	*n*	III

Begin by breaking the meter SSSISI into groups of threes (SSS-ISI). These groups correspond to the letters *mj*. At this point, you've essentially converted a binary number (base 2) into an octal number (base 8), which doesn't seem like much progress. However, Pingala had a very clever plan. The letters *mj* can be embedded in a word—say, "mojo"—that is more memorable that "number forty-one." For good measure, you write a poem in the "mojo" meter than describes the essential characteristics of the meter and includes the word "mojo." Musicians also use this method for remembering rhythm patterns.

This is perhaps the earliest example of an *error-correcting code*. Typically, an error-correcting code is a sequence that contains encoded information designed to flag errors in transmission (for example, typographical errors). Credit-card numbers and bar codes both have this feature. Each important aspect of a Sanskrit meter is encoded in the poem. In this case, the rhythm of the poem and the name of its meter provide a check on each other.

The history of the Sanskrit system of naming meters doesn't end with Pingala. Either in his time or later, the nonsense word *yamātārājabhānasalagā* came to be used as a way to remember the mapping of triplets of syllables to letters [7]. The word contains long and short syllables (in the English transliteration of Sanskrit, *a* is a short vowel and *ā* is a long vowel):

ya-mā-tā-rā-ja-bhā-na-sa-la-gām = ISSSISIIIS.

The pattern ISSSISIIIS has the curious property that each string of three syllables occurs *exactly* once. For example, the first three syllables form the pattern ISS, the second through fourth syllables form SSS, and so on. These patterns are named, using Pingala's table, by their starting syllable (so that *ya* represents ISS). However, the number of syllables in a meter doesn't have to be a multiple of three (you may have noticed this flaw in Pingala's method). The last two syllables, *la* and *gām* are used for the leftovers. It is not known whether Pingala knew this mnemonic for the triplets, or if it was discovered by poets and drummers that came after him.

The pattern ISSSISIIIS is very close to being what mathematicians call a *de Bruijn sequence*. A de Bruijn sequence is a sequence of letters drawn from some alphabet such that every combination of n letters occurs exactly once, if we are allowed to "wrap around" from the end of the sequence to the beginning. The string ISSSISII is a de Bruijn sequence (we get the combination III by wrapping around from the end to the beginning). Although SSSISIII is also a de Bruijn sequence, we don't think of these two as fundamentally different; they each produce the three-letter combinations in the same order, but start at a different point in the cycle. If we use this notion of sameness, there are only *two* possible de Bruijn sequences for three-letter patterns using the alphabet {S, I}. Figure 7 shows the pattern on a circle, alongside a de Bruijn cycle for four-letter patterns.

How can we find the other de Bruijn sequence for combinations of three letters? The other sequences of four letters? To investigate this question, it helps to start with an easier case:

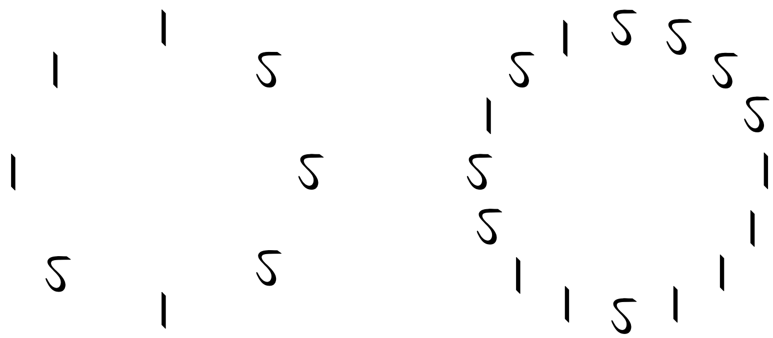

Figure 7: De Bruijn cycles for patterns of three and four letters

two letters. I'll leave it to you to show that there is only one de Bruijn sequence. One way to start is to write down the eight three-letter combinations. The de Bruijn sequence can be thought of as an ordering of these sequences: start with the first three letters in the sequence, then the second through fourth letters, etc. You probably notice that there are some rules about which of these can follow each other. For example, IIS can be followed by either ISI or ISS. In other words, either the string IISI or the string IISS must appear in your sequence. To find all the possibilities, we can use a powerful representation called a *directed graph*. The vertices of the graph represent states (in this case, three-letter combinations). If we can legally move from one state to another, connect them with an arrow. The graph is shown in figure 8. Any path that visits each vertex exactly once with give you a de Bruijn sequence.

The four-letter problem is more complex; you now have twice as many vertices. It is difficult to avoid drawing a graph that looks like a mound of spaghetti! There is an clever solution to this problem, however: represent each four-letter combination as an *edge* in the graph, as in Figure 9. In this case, you now need to find a path that visits each edge exactly once. I'll leave that to you.

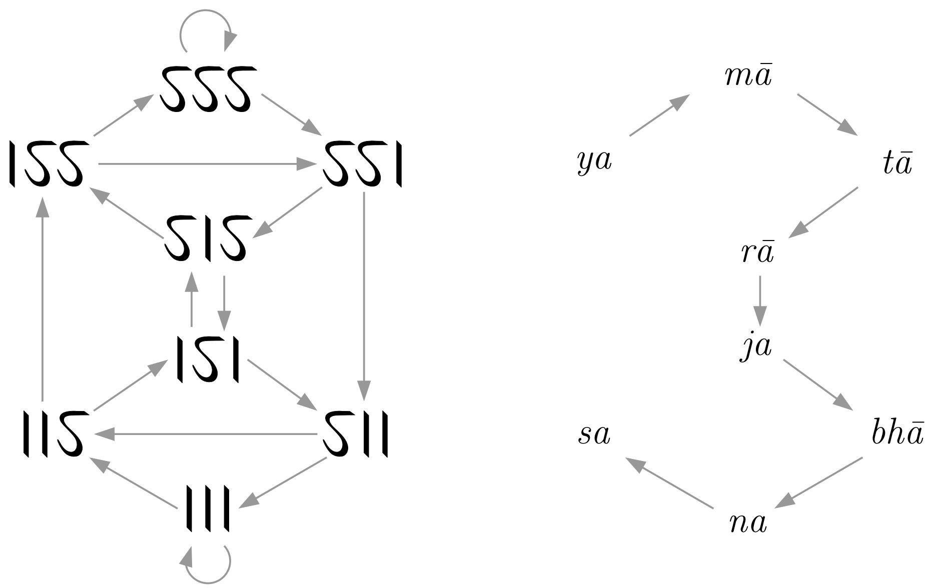

Figure 8: A graph representing the de Bruijn sequence problem, and a solution.

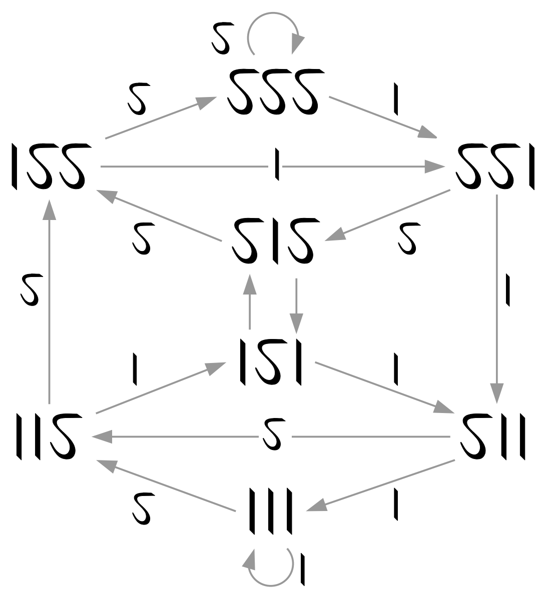

Figure 9: A graph representing the de Bruijn sequence problem for combinations of four letters.

Bibliography

[1] Amulya Kumar Bag. Binomial theorem in ancient India. *Indian J. History Sci.*, 1:68–74, 1966.

[2] A. W. F. Edwards. *Pascal's arithmetical triangle*. Johns Hopkins University Press, Baltimore, MD, 2002. The story of a mathematical idea, Revised reprint of the 1987 original.

[3] Georges Ifrah. *The universal history of numbers: from prehistory to the invention of the computer*. John Wiley & Sons Inc., New York, 2000. Translated from the 1994 French original by David Bellos, E. F. Harding, Sophie Wood and Ian Monk.

[4] Donald E. Knuth. *The art of computer programming. Vol. 4, Fasc. 4*. Addison-Wesley, Upper Saddle River, NJ, 2006. Generating all trees—history of combinatorial generation.

[5] Amba Kulkarni. Recursion and combinatorial mathematics in Chandashāstra, 2007. URL: http://arxiv.org/pdf/math/0703658. Retrieved June 1, 2007.

[6] David Luebbert. The rhythms on SongTrellis, 2005. URL: http://www.songtrellis.com/rhythmPage. Retrieved February 9, 2005.

[7] Curt Sachs. *Rhythm and Tempo*. W. W. Norton & Co., New York, 1953. A study in music history.

[8] Jeff Senn. Jas's – MIDI hand drum rhythm generator, 2005. URL: http://www.khafif.com/rhy/rhygen.html. Retrieved February 9, 2005.

[9] Parmanand Singh. The so-called Fibonacci numbers in ancient and medieval India. *Historia Math.*, 12(3):229–244, 1985.

[10] Ian Stewart. *Math Hysteria : Fun and games with mathematics*. Oxford University Press, Oxford, 2004.

[11] B. Van Nooten. Binary numbers in Indian antiquity. *Journal of Indian Philosophy*, 21:31–50, 1993.