Chapter Two

Mathematical Thought in Vedic India

2.1 THE VEDAS AND MATHEMATICS

As noted in section 1.3, the earliest extant Sanskrit texts are the ancient religious texts known as the Vedas, which are traditionally grouped into four *saṃhitās* or collections. Probably the oldest elements of these collections, based on comparisons of their vocabulary and grammatical and prosodic forms, are hymns to various deities in some sections of the *Ṛg-veda* or "Praise-Knowledge." The standard model of ancient Indian historiography places their composition sometime in the second millennium BCE. Somewhat later than these Early Vedic hymns are Middle Vedic invocations or mantras used in rituals for performing religious sacrifices, recorded in the *Yajur-veda* ("Sacrifice-Knowledge"). The other two Vedic collections are the *Sāma-veda* ("Chant-Knowledge") and the *Atharva-veda* ("Knowledge of the Atharvan-priest"), containing chants, prayers, hymns, curses, and charms.

This knowledge was shaped into a canonical corpus probably sometime before the middle of the first millennium BCE. The remaining works identified as part of *śruti* or revealed wisdom were composed to interpret and expound the Vedas. Among these, the Brāhmaṇa texts chiefly describe and explain sacrificial ritual. (These texts are not to be confused with the human Brāhmaṇas, or "Brahmins," who were hereditary priests and scholars.) The compositions called Vedānta, or "end of the Vedas," comprising the Āraṇyakas and Upaniṣads, contain teachings on philosophical and spiritual themes.

What do these texts tell us about ancient Indian ideas on mathematical subjects? In the first place, they reveal that by Early Vedic times a regularized decimal system of number words to express quantity was well established. (Most of these number words evidently date back as far as Proto-Indo-European, since they have many cognates in other Indo-European languages.) Some of the most archaic Vedic hymns attest to this system based on decades and powers of ten, including combined numbers involving both decades and units:

> You, radiant [Agni, the fire-god], are the lord of all [offerings]; you are the distributor of thousands, hundreds, tens [of good things]. (*Ṛg-veda* 2.1.8)

> Come, Indra [king of the gods], with twenty, thirty, forty horses; come with fifty horses yoked to your chariot, with sixty, seventy, to drink the [sacred beverage] Soma; come carried by eighty, ninety, a hundred horses. (*Ṛg-veda* 2.18.5–6)
>
> Three thousand three hundred and thirty-nine [literally "three hundreds, three thousands, thirty and nine"] gods have worshipped Agni ... (*Ṛg-veda* 3.9.9)

Some simple fractional parts such as one-third, using ordinal number forms as in their English equivalents, also occur in Early or Middle Vedic texts.[1] No later than the Middle Vedic period the Indian decimal integers had been expanded to a remarkable extent with the addition of number words for much larger powers of ten, up to at least a trillion (10^{12}). The first record of them occurs among the hymns included in the *Yajur-veda*'s descriptions of sacrificial rites. These hymns invoke not only deities but also aspects of nature and abstract entities, including various sequences of numbers, both round and compound:

> Hail to earth, hail to the atmosphere, hail to the sky, hail to the sun, hail to the moon, hail to the *nakṣatras* [lunar constellations], hail to the eastern direction, hail to the southern direction, hail to the western direction, hail to the northern direction, hail to the upwards direction, hail to the directions, hail to the intermediate directions, hail to the half-years, hail to the autumns, hail to the day-and-nights, hail to the half-months, hail to the months, hail to the seasons, hail to the year, hail to all. (*Yajur-veda* 7.1.15)
>
> Hail to one, hail to two, hail to three ... hail to eighteen, hail to nineteen [literally "one-less-twenty"], hail to twenty-nine [literally "nine-twenty"], hail to thirty-nine ... hail to ninety-nine, hail to a hundred, hail to two hundred, hail to all. (*Yajur-veda* 7.2.11)
>
> Hail to a hundred, hail to a thousand, hail to *ayuta* [ten thousand], hail to *niyuta* [hundred thousand], hail to *prayuta* [million], hail to *arbuda* [ten million], hail to *nyarbuda* [hundred million], hail to *samudra* [billion], hail to *madhya* [ten billion], hail to *anta* [hundred billion], hail to *parārdha* [trillion], hail to the dawn, hail to the daybreak ... hail to the world, hail to all. (*Yajur-veda* 7.2.20)

Why did Vedic culture construct such an extensive number system and acclaim it in sacred texts? The computing requirements of everyday life

[1]For example, in *Yajur-veda* 2.4.12.3: "He, Viṣṇu, set himself in three places, a third on the earth, a third in the atmosphere, a third in the sky." (All the *Yajur-veda* cites in this chapter are from the version known as the Taittirīya-saṃhitā recension of the Kṛṣṇa Yajurveda. Note that in these quoted passages and all others throughout the book, text in square brackets represents editorial additions and explanations that are not literally present in the original.)

would not have demanded more than the first few decimal orders of magnitude, as seen among other ancient civilizations, whose known number words reach only into the thousands or tens of thousands. Although infinite speculations are possible about the metaphysical or spiritual implications of these numbers in Vedic thought, there is probably no conclusive solution to the mystery.[2]

The cosmic significance of numbers and arithmetic in ritual reflecting concepts of the universe is brought out clearly in another early first-millennium text, the *Śata-patha-brāhmaṇa* or "Brāhmaṇa of a hundred paths," an exegetical text explaining the symbolism of sacrificial rituals. The following passage refers to sacrificial fire-altars made of baked bricks which symbolize the 720 days and nights of an ideal year. The creator god Prajāpati, representing this year and the concept of time in general, sought to regain power over his creation by arranging these 720 bricks in various ways:

> Prajāpati, the year, has created all existing things.... Having created all existing things, he felt like one emptied out, and was afraid of death. He bethought himself, "How can I get these beings back into my body?"... He divided his body into two; there were three hundred and sixty bricks in the one, and as many in the other; he did not succeed. He made himself three bodies.... He made himself six bodies of a hundred and twenty bricks each; he did not succeed. He did not divide sevenfold. He made himself eight bodies of ninety bricks each.... He did not divide elevenfold.... He did not divide either thirteenfold or fourteenfold.... He did not divide seventeenfold. He made himself eighteen bodies of forty bricks each; he did not succeed. He did not divide nineteenfold. He made himself twenty bodies of thirty-six bricks each; he did not succeed. He did not divide either twenty-onefold, or twenty-twofold, or twenty-threefold. He made himself twenty-four bodies of thirty bricks each. There he stopped, at the fifteenth; and because he stopped at the fifteenth arrangement there are fifteen forms of the waxing, and fifteen of the waning [moon]. And because he made himself twenty-four bodies, therefore the year consists of twenty-four half-months....[3]

(The full sequence of attempted or rejected divisions by all the integers from 2 to 24 is described in the text, although the above excerpt omits some of them for conciseness.)

The final division of 720 into 24×30 is the last possible one that will give an integer quotient. Even more interesting, mathematically speaking, than Prajāpati's ultimate successful division are the divisions that he did

[2] Some inferences about the mystical meaning of numbers are discussed in [BerA1878], vol. 2, ch. 5, in [Mal1996], ch. 14, and in [Mur2005].

[3] *Śata-patha-brāhmaṇa* 10.4.2, [Egg1897], pp. 349–351. I have substituted "divide" as the translation of *vi-bhū* where Eggeling uses "develop." See also the discussion in [Mal1996], ch. 13.

not attempt, which would have produced fractional numbers of bricks. The concept of integer divisibility is thus part of this cosmic narrative. Its sequence of pairs of factors of 720, with the numbers relatively prime to 720 neglected, somewhat resembles Old Babylonian tables of sexagesimal reciprocals or paired factors of the base 60, where 2 is coupled with 30, 3 with 20, and so on, while the relatively prime numbers such as 7 and 11 are omitted.[4] The sexagesimal multiple 720 is also familiar in Old Babylonian texts, being the standard metrological unit called the "brick-sar."[5] Whether these similarities are the result of coincidence or hint at some kind of early transmission remains unclear. Most of the chief characteristic features of Old Babylonian mathematics—sexagesimal place-value numbers, tables for multiplication and division, written numeral forms—have no counterpart in the scanty available evidence for Vedic mathematical ideas.

Late Vedic exegetical texts such as the Upaniṣads, as well as contemporary Buddhist and Jaina philosophy, also offer intriguing possibilities for speculation about the development of some concepts later incorporated in mathematics per se. Examples of these include the synonyms *śūnya* and *kha*, meaning "void," "nullity" (in later mathematical texts "zero") and *pūrṇa* or "fullness."[6] Unfortunately, we have no distinct lines of textual descent from Vedic religious and philosophical compositions on such concepts to their later embodiment in specifically mathematical works. About all we can say is that the Vedic texts clearly indicate a long-standing tradition of decimal numeration and a deep fascination with various concepts of finite and infinite quantities and their significance in the cosmos.[7]

2.2 THE *ŚULBA-SŪTRAS*

Mathematical ideas were explored in more concrete detail in some of the ancillary works classified as Vedāṅgas, "limbs of the Vedas," mentioned in section 1.3—phonetics, grammar, etymology, metrics, astronomy and calendrics, and ritual practice. This section examines mathematics in Vedāṅga

[4] [Hoy2002], pp. 27–30.

[5] [Rob1999], p. 59.

[6] See, for example, [Gup2003] and [Mal1996], ch. 3.

[7] Popular usage of the term "Vedic mathematics" often differs considerably from the mathematical content actually attested in Vedic texts. Some authors use "Vedic mathematics" to mean the entire Sanskrit mathematical tradition in Vedic and post-Vedic times alike, which of course comprises much more than is directly present in these early sources. Most commonly, though, the term signifies the Sanskrit mental-calculation algorithms published in 1965 in a book entitled *Vedic Mathematics*, which the author described ([Tir1992], pp. xxxiv–xxxv) as "reconstructed from" the *Atharva-veda* and which are very popular nowadays in mathematics pedagogy. These algorithms are not attested in any known ancient Sanskrit text and are not mentioned in traditional Vedic exegesis. They constitute an ingenious modern Sanskrit presentation of some mathematical ideas rather than an ancient textual source. The widespread confusion on this topic has been addressed in [DanS1993] and [SarS1989], and a thorough scrutiny of the explicitly mathematical and numerical references that actually appear in the four Vedic collections is presented in [Pandi1993].

texts on ritual practice, which specified the details of performing the various ceremonies and sacrifices to the gods. These texts were classified either as pertaining to *śruti* and describing major ceremonies, or as pertaining to *smṛti* and explaining the routine customs and observances to be maintained in individual households. The former type included the regular fire sacrifices performed at particular times of the year and the month, as well as special rituals sponsored by high-ranking individuals for particular aims, such as wealth, military victory, or heaven in the afterlife.

Some of the ritual practice texts explained how the different types or goals of sacrifices were associated with different sizes and shapes of fire altars, which were to be constructed from baked bricks of prescribed numbers and dimensions. The footprints for the altars were laid out on leveled ground by manipulating cords of various lengths attached to stakes. The manuals described the required manipulations in terse, cryptic phrases—usually prose, although sometimes including verses—called *sūtras* (literally "string" or "rule, instruction"). The measuring-cords, called *śulba* or *śulva*, gave their name to this set of texts, the *Śulba-sūtras*, or "Rules of the cord."

Many of the altar shapes involved simple symmetrical figures such as squares and rectangles, triangles, trapezia, rhomboids, and circles. Frequently, one such shape was required to be transformed into a different one of the same size. Hence, the *Śulba-sūtra* rules often involve what we would call area-preserving transformations of plane figures, and thus include the earliest known Indian versions of certain geometric formulas and constants.

How this ritual geometry became integrated with the process of sacrificial offerings is unknown. Did its mathematical rules emerge through attempts to represent cosmic entities physically and spatially in ritual?[8] Or conversely, was existing geometric knowledge consciously incorporated into ritual practice to symbolize universal truth or to induce a "satori" state of mind in the participants through perception of spatial relationships? No contemporary text can decide these questions for us: the concise *Śulba-sūtras* themselves are mostly limited to essential definitions and instructions, and the earliest surviving commentaries on them are many centuries later than the *sūtras*, which in turn are doubtless later than the mathematical knowledge contained in them.

The rest of the historical context of the *Śulba-sūtras* is also rather vague. The ritual practice text corpora to which they belong are ascribed to various ancient sages about whom no other information survives. The best-known *Śulba-sūtras* are attributed to authors named Baudhāyana, Mānava, Āpastamba, and Kātyāyana, in approximately chronological order. They are assigned this order on the basis of the style and grammar of the language of their texts: those of Baudhāyana and Mānava seem to be roughly contempo-

[8]This is the hypothesis of, for example, [Sei1978], in which a prehistoric ritual origin for Eurasian geometry traditions is reconstructed from ideas of the sky as a circle, the earth as a square, and so on. And [Sta1999] amplifies this thesis for a potential Indo-European ancestor of both Indian and Greek geometry, based on the ritual associations of both *Śulba-sūtra* techniques and the "altar of Delos" legend of the cube duplication problem.

rary with Middle Vedic Brāhmaṇa works composed perhaps in 800–500 BCE, while the *Śulba-sūtra* of Kātyāyana appears to post-date the great grammatical codification of Sanskrit by Pāṇini in probably the mid-fourth century BCE. Nothing else is known, and not much can be guessed, about the lives of these texts' authors or the circumstances of their composition.[9]

The *Śulba-sūtras*, like other manuals on ritual procedure, were intended for the use of the priestly Brāhmaṇa families whose hereditary profession it was to conduct the major sacrificial rituals. But since animal sacrifice and consequently most of the fire altar rituals were eventually abandoned in mainstream Indian religion, and since there are few archaeological traces of ancient fire altars, it is not certain how the prescribed procedures were typically enacted in practice.[10]

The *Śulba-sūtra* texts[11] include basic metrology for specifying the dimensions of bricks and altars. Among the standard units are the *aṅgula* or digit (said to be equal to fourteen millet grains), the elbow-length or cubit (twenty-four digits), and the "man-height" (from feet to upraised hands, defined as five cubits).[12] As early as the *Baudhāyana-śulba-sūtra*, methods are described for creating the right-angled corners of a square or rectangle, constructing a square with area equal to the sum or difference of two given squares, and transforming a square with area preservation into a rectangle (or vice versa), into a trapezium, triangle or rhombus, or into a circle (or vice versa). In the process, it is explicitly recognized that the square on the diagonal of a given square contains twice the original area; and more generally that the squares on the width and the length of any rectangle add up to the square on its diagonal (the so-called Pythagorean theorem).[13] Samples

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[9] See [SenBa1983], pp. 2–5. It is suggested in [Pin1981a], pp. 4–5, that the Āpastamba and Kātyāyana *Śulba-sūtras* predate that of Mānava. In [Kak2000a], a much earlier date for *Śulba-sūtra* works is inferred by linking them to astrochronological speculations (see section 2.3).

[10] An archaeological site containing one large brick altar in the traditional shape of a bird with outstretched wings, but differing markedly from the numerical specifications described in the *Śulba-sūtra* texts, has been dated to the second century BCE; [Pin1981a], p. 4, n. 19. And a long-lived South Indian tradition of fire altar construction is attested at the present day in [Sta1983] and in [Nam2002]. But since both of these may have originated in a form of "Vedic revivalism" in some post-Vedic period rather than in a continuous ritual praxis going back to the composition of the *Śulba-sūtras*, we cannot be sure how far either of them represents the original tradition of fire-altar geometry. In [SarE1999], pp. 10–11, such a lapse and revival in the abovementioned South Indian ritual tradition after about the fourth century CE are mentioned.

[11] For an edition and annotated English translation of the four major *Śulba-sūtra* works, see [SenBa1983], on whose edition the following translations are based. Sūtras 1.1–1.2, 1.4–1.13, and 2.1–2.12 of the *Baudhāyana-śulba-sūtra* are quoted and commented on in [Plo2007b], pp. 387–393. An earlier study of *Śulba-sūtra* mathematics is [Dat1993].

[12] See the various metrological *sūtras* in *Baudhāyana-śulba-sūtra* 1.3, [SenBa1983], pp. 17 (text), 77 (translation); *Mānava-śulba-sūtra* 4.4–6, [SenBa1983], pp. 60, 128; *Āpastamba-śulba-sūtra* 15.4, [SenBa1983], pp. 49, 113; *Kātyāyana-śulba-sūtra* 5.8–9, [SenBa1983], pp. 57, 124.

[13] *Baudhāyana-śulba-sūtra* 1–2; [SenBa1983], pp. 17–19 (text), 77–80 (translation). Henceforth the *Śulba-sūtra* citations will be confined to identifying the text and *sūtra* in the edition of [SenBa1983]. The abbreviations used for the text names are listed on

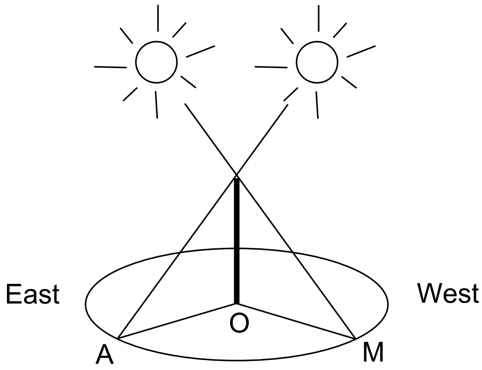

Figure 2.1 Determining the east-west line with shadows cast by a stake.

of such rules from various *Śulba-sūtra* texts are cited in the following part of this section, along with some of their procedures for more elaborate altar constructions.

The preliminary step is the drawing of a baseline running east and west. We do not know for sure how this was accomplished in the time of the early *Śulba-sūtra* authors, but the later *Kātyāyana-śulba-sūtra* prescribes using the shadows of a gnomon or vertical rod set up on a flat surface, as follows:

> Fixing a stake on level [ground and] drawing around [it] a circle with a cord fixed to the stake, one sets two stakes where the [morning and afternoon] shadow of the stake tip falls [on the circle]. That [line between the two] is the east-west line. Making two loops [at the ends] of a doubled cord, fixing the two loops on the [east and west] stakes, [and] stretching [the cord] southward in the middle, [fix another] stake there; likewise [stretching it] northward; that is the north-south line. (*KāSS* 1.2)

The first part of the procedure is illustrated in figure 2.1, where the base of the gnomon is at the point O in the center of a circle drawn on the ground.[14] At some time in the morning the gnomon will cast a shadow OM whose tip falls on the circle at point M, and at some time in the afternoon the gnomon will cast a shadow OA that likewise touches the circle. The line between points A and M will run approximately east-west.

Then a cord is attached to stakes at the east and west points, and its midpoint is pulled southward, creating an isosceles triangle whose base is the east-west line. Another triangle is made in the same way by stretching the cord northward. The line connecting the tips of the two triangles is a perpendicular bisector running north and south. Similar ways of stretching

page xiii.

[14]Note that the text itself is purely verbal and contains no diagrams. This figure and all the remaining figures and tables in this chapter are just modern constructs to help explain the mathematical rules.

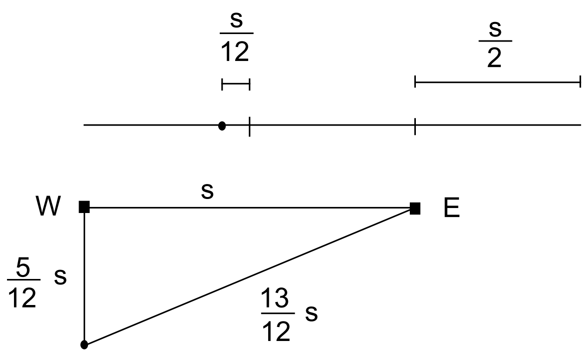

Figure 2.2 Determining the perpendicular sides of a square with a marked cord.

a cord into a triangle are also used for basic determinations of right-angled figures, as in the following construction of a square:

> The length is as much as the [desired] measure; in the western third of [that length] increased by its half, at the [place] less by a sixth part [of the third], one makes a mark. Fastening [the ends of the cord] at the two ends of the east-west line, stretching [the cord] southward by [holding] the mark, one should make a marker [at the point that it reaches]. In the same way [one should stretch the cord] northward; and in the other two directions after reversing [the ends of the cord]. That is the determination. [There is] shortening or lengthening [of the side to produce the desired half-side of the square with respect to] that marker. (*ĀpSS* 1.2)

Here a cord with length equal to the desired side of a square, say s, is increased to a total length of $\frac{3}{2}s$, and a mark is made at a distance of $\frac{5}{12}s$ from one end, as shown in figure 2.2. So when the endpoints are fixed a distance s apart along the east-west line, pulling the mark downwards creates a 5-12-13 right triangle to make the sides perpendicular. The same technique is also used with 3-4-5 right triangles (e.g., in *BauSS* 1.5, *KāSS* 1.4). More general properties of sides and diagonals are stated as well, including versions of what we now call the Pythagorean theorem and a rule for the length of the diagonal of a square with a given "measure" or side:

> The cord [equal to] the diagonal of an oblong makes [the area] that both the length and width separately [make]. By knowing these [things], the stated construction [is made]. (*ĀpSS* 1.4; similarly *BauSS* 1.12)
>
> The cord [equal to] the diagonal of a [square] quadrilateral makes twice the area. It is the doubler (*dvi-karaṇī*, "two-maker") of the

> square. (*ĀpSS* 1.5; similarly *BauSS* 1.9, *KāSS* 2.8)
>
> One should increase the measure by a third [part] and by a fourth [part] decreased by [its] thirty-fourth [part]; [that is its] diagonal [literally "together-with-difference"]. (*ĀpSS* 1.6; similarly *BauSS* 2.12, *KāSS* 2.9)

This rule for the length of the diagonal of a square of side s equates it to $s\left(1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}\right)$, or about $s \times 1.4142$. Interestingly, the *Kātyāyana-śulba-sūtra* version calls this rule approximate or "having a difference" (from the exact value).

Areas involving multiples of three are also constructed. For example, if a rectangle is made with width equal to the original square side s and length equal to its "doubler" or $\sqrt{2}s$, then the diagonal of the rectangle is declared to be the "tripler," producing a square of three times the original area:

> The measure is the width, the doubler is the length. The cord [equal to] its hypotenuse is the tripler (*tri-karaṇī*). (*ĀpSS* 2.2; similarly *BauSS* 1.10, *KāSS* 2.10)
>
> The one-third-maker (*tṛtīya-karaṇī*) is explained by means of that. [It is] a ninefold division [from the square on the tripler]. (*ĀpSS* 2.3; similarly *BauSS* 1.11, *KāSS* 2.11)

That is, an area one-third of the original area will be one-ninth of the square on the tripler.

Some typical transformations of one figure into another are the following procedures for "combining" or "removing" squares, that is, adding or subtracting square areas:

> The combination of two equal [square] quadrilaterals [was] stated. [Now] the combination of two [square] quadrilaterals with individual [different] measures. Cut off a part of the larger with the side of the smaller. The cord [equal to] the diagonal of the part [makes an area which] combines both. That is stated. (*ĀpSS* 2.4; similarly *BauSS* 2.1, *KāSS* 2.13)
>
> Removing a [square] quadrilateral from a [square] quadrilateral: Cut off a part of the larger, as much as the side of the one to be removed. Bring the [long] side of the larger [part] diagonally against the other [long] side. Cut off that [other side] where it falls. With the cut-off [side is made a square equal to] the difference. (*ĀpSS* 2.5; similarly *BauSS* 2.2, *KāSS* 3.1)

The first of these *sūtras* begins by noting that the previously given definition of the "doubler" or diagonal of a square in essence explained how to make a square equal to the sum of two identical squares. The methods for adding and subtracting two squares of different sizes, again relying on the relations between the sides and hypotenuses of right triangles, are illustrated

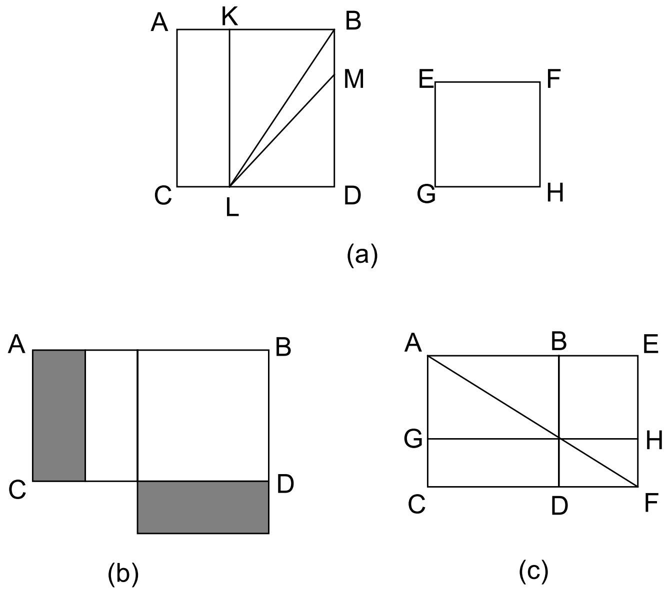

Figure 2.3 Transformations of squares and rectangles.

in figure 2.3a. If $ABCD$ is the larger square and $EFGH$ the smaller, cut off from $ABCD$ a rectangle $KBLD$ with width equal to the shorter side and length equal to the longer. Then its diagonal LB will be the side of a square equal to the sum of the two given squares. But if instead the long side KL is placed diagonally as the segment LM, then the cut-off side MD will be the side of a square equal to their difference. This second technique is employed again in transforming a rectangle into a square:

> Wishing [to make] an oblong quadrilateral an equi-quadrilateral: Cutting off [a square part of the rectangle] with [its] width, [and] halving the remainder, put [the halves] on two [adjacent] sides [of the square part]. Fill in the missing [piece] with an extra [square]. Its removal [has already been] stated. (*ĀpSS* 2.7; similarly *BauSS* 2.5, *KāSS* 3.2)

> Wishing [to make] an equi-quadrilateral an oblong quadrilateral: Making the length as much as desired, put whatever is left over where it fits. (*ĀpSS* 3.1; similarly *BauSS* 2.4)

In the first of these two rules, as shown in figure 2.3b, a square with side BD equal to the width of the given rectangle $ABCD$ is cut off from it, and the remainder of the rectangle is divided into two halves, one of which (shaded in the figure) is placed on the adjacent side of the square. This produces an L shape (also called a gnomon figure—no relation to the vertical stick gnomon for casting shadows) with an empty corner that will have to be filled in with an additional square piece, but the desired square side can then be found by the square-subtraction procedure described above.

It is not quite clear what the *śulba*-priest is supposed to do in the converse case of converting a square into a rectangle. It seems as though a rectangle of the desired width is to be cut off from the square and the remaining bricks of the square's area packed onto the rectangle's end in an ad hoc way. Later commentators have suggested a more rigorous interpretation,[15] illustrated in figure 2.3c, where the given square $ABCD$ is expanded into a rectangle $AECF$ of the desired length AE. Then the intersection of the diagonal AF with the original square side BD defines the side GH of the required rectangle $AEGH$ with area equal to that of the original square. However, this does not seem to be what the *sūtra* actually says, although it is somewhat similar to a simpler transformation rule (*BauSS* 2.3, *KāSS* 3.4) where a square of side s is cut diagonally into three triangles—one half and two quarters—with the quarters then shifted to form a rectangle with dimensions $s\sqrt{2} \times \frac{s\sqrt{2}}{2}$.

Transformations between rectilinear and circular shapes are also tackled:

> Wishing to make a [square] quadrilateral a circle: Bring [a cord] from the center to the corner [of the square]. [Then] stretching [it] toward the side, draw a circle with [radius equal to the half-side] plus a third of the excess [of the half-diagonal over the half-side]. That is definite[ly] the [radius of the] circle. As much as is added [to the edges of the circle] is taken out [of the corners of the square]. (*ĀpSS* 3.2; similarly *BauSS* 2.9, *KāSS* 3.11)
>
> Wishing [to make] a circle a [square] quadrilateral: Making the diameter fifteen parts, remove two. Thirteen [parts] remain. That is indefinite[ly, approximately] the [side of the square] quadrilateral. (*ĀpSS* 3.3; similarly *BauSS* 2.11, *KāSS* 3.12)

In the first of these *sūtras*, the radius of a circle with area equal to a given square is taken to be the half-side of the square, plus one-third of the difference between the half-side and the half-diagonal; that is, the radius is said to equal $\frac{s}{2} + \frac{s\sqrt{2}/2 - s/2}{3}$. To convert instead a given circle into a desired square, one is supposed to use $\frac{13}{15}$ of the diameter of the circle as the square's side; but this is apparently not considered as accurate as the first formula. (See the list in table 2.1 at the end of this section for a comparison of the different values of constants implied by these rules.)

Let us now look at the *Śulba-sūtra* specifications for some actual altar arrangements, starting with the prescribed setup of the traditional three fires used for most sacrificial ceremonies. These are the "householder's fire," which must burn continually under the care of each individual householder,

[15]See [SenBa1983], pp. 156–158.

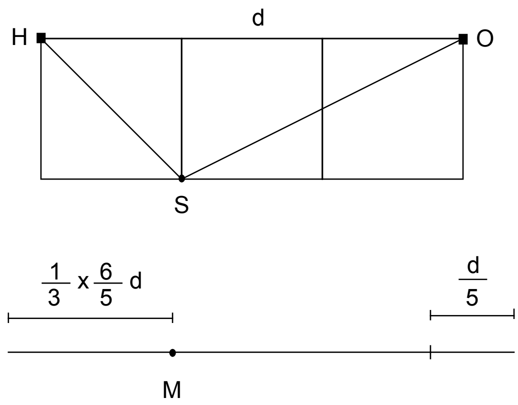

Figure 2.4 Laying out the three sacrificial fires.

the "oblation fire," and the "southern fire." They are to be arranged as follows:

> Now in the construction for setting up the [sacrificial] fires, the distance from the householder's to the oblation [fire]. It is known: The Brāhmaṇa sets [the latter] fire at eight double-paces [where a pace equals 15 *aṅgulas*], the prince eleven, the Vaiśya twelve, [east of the householder's fire]. (*BauSS* 3.1; similarly *ĀpSS* 4.1)
>
> Make three successive [contiguous square] quadrilaterals with [sides equal to] a third of [that] length. In the northwest corner is the householder's [fire]. In the south[east] corner of that same [square] is the [southern] offering fire; in the northeast corner [of the whole] is the oblation [fire]. (*BauSS* 3.2; similarly *ĀpSS* 4.3)

The three fires form a triangle as shown in figure 2.4, with the householder's and oblation fires (H and O respectively) at the western and eastern ends respectively of the east-west line HO; the length of HO depends on the rank of the sacrificer (see section 6.1.2 for a description of the ranks alluded to). The place of the southern fire S (south of the line, as its name suggests) is to be found by laying out the required three squares in a row south of HO. Then S is set in the southeast corner of the western square.

Or else, according to the texts, one can approximate this layout by means of a stretched-cord construction, as follows:

> Dividing the distance [between] the householder's and the oblation [fires] into five or six parts, adding an extra sixth or seventh part, dividing the whole into three, making a mark at the western third, fastening [the ends] at the householder's and oblation [fires and] stretching [the cord] southward by [holding] the mark,

> one should make a marker. That is the place of the southern fire. It agrees with *smṛti*. (*ĀpSS* 4.4; similarly *BauSS* 3.3)

The prescribed cord is also shown in figure 2.4. There it has length $\frac{6}{5}d$, where d is the distance HO between the first two fires (the user may instead choose to make the length equal to $\frac{7}{6}d$). The cord is then divided into three equal parts, and a mark M is made at the eastern end of the western third, that is, at a distance of $\frac{1}{3} \cdot \frac{6}{5}d = \frac{2}{5}d$ (or alternatively $\frac{1}{3} \cdot \frac{7}{6}d = \frac{7}{18}d$) from the western end of the cord.

When the marked cord is attached at the endpoints H and O and stretched toward the south, the mark M is supposed to fall approximately at S, the place of the southern fire. Of course, since the marked length $\frac{2}{5}d$ is somewhat shorter than the actual diagonal of the square $HS = \frac{\sqrt{2}}{3}d$, the triangle produced by the cord will not be exactly congruent to HOS.

An important related construction is that of the Great Altar or "*soma*-sacrifice altar" used in the ceremonies of the sacred ritual beverage *soma* (see section 1.2). The Great Altar is to be set up east of the three fires in the shape of an isosceles trapezium with its base facing west, using prescribed dimensions:

> [The altar] is thirty paces or double-paces on the western side, thirty-six on the east-west line, twenty-four on the eastern side: thus the [dimensions] of the *soma*-altar are known. (*ĀpSS* 5.1; similarly *BauSS* 4.3)

> Adding eighteen [units] to a length of thirty-six, [making] a mark at twelve [units] from the western end [and another] mark at fifteen, fastening [the ends of the cord] at the ends of the east-west line, stretching [the cord] south by [holding] the fifteen [mark], one fixes a stake [there]; in the same way northward; those are the two [western] corners. Reversing the two ends, stretching [the cord] by [holding] the same fifteen [mark], one fixes a stake at the twelve [mark]. In the same way northward; those are the two [eastern] corners. That is the construction with one cord. (*ĀpSS* 5.2)

The Great Altar is to be laid out symmetrically about the east-west line as shown in figure 2.5 by means of the now familiar stretched-cord method, utilizing a 15-36-39 right triangle. The height of the trapezium $ABCD$, thirty-six units, is paced off along the east-west line, and its base AB of thirty units is found by stretching the cord twice, to the south and to the north, to form the right triangles WAE and WBE. The same procedure is performed on the eastern side, and the twelve-unit lengths ED and EC are marked off to form the trapezium's top CD.

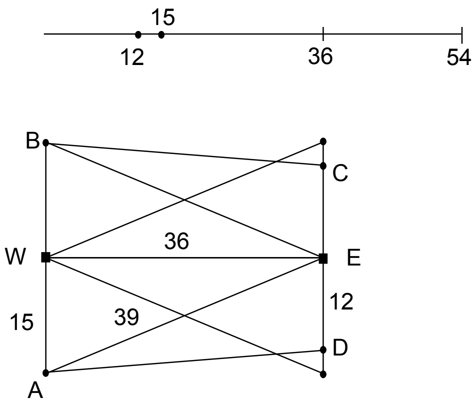

Figure 2.5 Construction of the trapezoidal Great Altar.

The text then describes how to cut and paste this figure into a rectangle—apparently just a mental construction for determining its area of 972 square units:

> The Great Altar is a thousand [square] paces [or double-paces] less twenty-eight. One should bring [a line] from the south[east] corner twelve units toward the south[west] corner. One should place the cut-off [triangle] upside-down on the other [side]. That is an oblong quadrilateral. In that way one should consider it established. (*ĀpSS* 5.7)

For a special sacrifice to the chief of the gods, Indra, the ritual requires a smaller altar with identical proportions to this Great Altar but only one-third of its area. To achieve the desired figure, the linear unit in the Great Altar construction is replaced by its "one-third-maker" (or $\frac{\sqrt{3}}{3}$) described above. Or else the altar dimensions are stated as smaller multiples of the unit's "tripler," $\sqrt{3}$:

> One should sacrifice with one-third [the area] of the *soma*-altar: this is known [for the area] of the Indra-sacrifice altar. The one-third-maker of the double-pace is to be used in place of the double-pace. Or, the widths [eastern and western sides] will be eight [and] ten [times] the tripler, [and] the east-west [length] twelve [times]. The Indra-sacrifice altar is three hundred twenty-four [square] paces [or double-paces]. (*ĀpSS* 5.8; similarly *BauSS* 3.12)

What does this ritual geometry add to our understanding of ancient Indian mathematical thought? For one thing, we see that at least by the time of the *Baudhāyana-śulba-sūtra*, arithmetic (although still not attested

Table 2.1 *Śulba-sūtra* constants

Sūtra	Rule and modern equivalent	Remarks; value
BauSS 2.9, *MāSS* 1.8, *ĀpSS* 3.2, *KāSS* 3.11	Half diagonal of square, minus difference of half diagonal and half side, plus one-third that difference, is radius of circle: $r = \frac{s}{2} + \frac{s\sqrt{2}/2 - s/2}{3}$	$\pi \approx 3.08831$
BauSS 2.10	Seven-eighths diameter of circle, plus one twenty-ninth of remaining eighth, minus one sixth of that twenty-ninth diminished by its eighth, is side of square: $s = \frac{2r}{8}\left(7 + \frac{1}{29} - \left(\frac{1}{29 \cdot 6} - \frac{1}{29 \cdot 6 \cdot 8}\right)\right)$	$\pi \approx 3.08833$
BauSS 2.11, *ĀpSS* 3.3, *KāSS* 3.12	Thirteen-fifteenths of diameter of circle is side of square: $s = 2r \cdot \frac{13}{15}$	Called "approximate" $\pi \approx 3.004$
BauSS 2.12, *ĀpSS* 1.6, *KāSS* 2.9	Side of square plus its third plus a fourth of the third minus one thirty-fourth of the fourth is the diagonal $s\sqrt{2} = s \cdot \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\right)$	*KāSS* says "approximate" $\sqrt{2} \approx 1.4142$
MāSS 11.9–10	$s^2 = \frac{3(2r)^2}{4}$	So interpreted in [Hay1990] $\pi \approx 3$
MāSS 11.15	$r = \frac{4}{5} \cdot \frac{\sqrt{2}}{2} s$	So interpreted in, e.g., [Gup2004b] $\pi \approx 3.125$

in written form) embraced manipulation of arbitrary fractional parts such as one thirty-fourth or two fifteenths, sometimes in quite complicated combinations. Spatial properties of several rectilinear plane figures were well understood, including the relationships among their sides, diagonals, and areas. Properties of the circle were also studied, particularly the challenging task of transforming it into a square of equal area or vice versa. And it was recognized that some of these transformation methods were more accurate than others in terms of preserving area. The transformation rules in fact corresponded to what we would call different values of irrational constants; several of them are summarized in table 2.1.

We have to be cautious about inferring any clear line of chronological development for any of these formulas, since it is perfectly possible that a later text could preserve an archaic rule that was omitted from an earlier text. We are also hampered by the textual isolation of these rules in efforts to understand how they were interpreted, derived or justified by their users. Modern scholars have suggested many ingenious ways to reconstruct their creation and explore their possible implications for other areas of mathematical thought.[16] But none of these is explicitly confirmed by the texts themselves, and there are no known textual traditions directly linking them to extant later works on geometry, which were composed starting around the middle of the first millennium CE. Nonetheless, as we will see in our exploration of those works in chapter 5, we frequently seem to hear in the verses of Classical Sanskrit geometry echoes of the *sūtras* of the ancient *śulba*-priests.

2.3 THE VEDAS AND ASTRONOMY

It has long been debated whether the Vedic corpus, in addition to providing clues about general numeration practices and ritual geometry, also preserves information about an ancient Indian tradition of mathematical astronomy.[17] Since later Sanskrit mathematics is so often closely tied to astronomical texts, it would not be surprising if we found the two subjects linked in Vedic times as well. Certainly there are clear references in Vedic texts to some astronomical and chronometric concepts, as illustrated by one of the Vedic hymns, quoted in section 2.1, which praises not only the sun, moon and constellations but also the directions, seasons, and months.

Vedic texts prescribed periodic sacrifices to be performed at particular times, such as the new and full moon, solstices, and equinoxes. This required keeping track of the passage of seasons and synodic months (synodic

[16]For example, see, in addition to the references in table 2.1, [Del2005] and [SenBa1983], pp. 165–168, for intriguing derivations of the $\sqrt{2}$ value in *Baudhāyana-śulba-sūtra* 2.12/*Āpastamba-śulba-sūtra* 1.6, particularly the geometric reconstruction following [Dat1993], pp. 192–194; [Neu1969], p. 34, for speculation on a possible relationship of this value to Old Babylonian mathematics; and [Knu2005] for square root rules and their possible connections to general quadratic problems.

[17]In this section the reader may wish to refer to the glossary in section 4.1 for explanations of unfamiliar astronomical terms.