Mathematics

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Mathematics in India of the Vedic age

The Focus of this paper - by R. L. Kashyap (Professor of Electrical and Computer engineering, Purdue University, West Lafayette, IN 47907, e-mail: kashyap @ ecn.purdue.edu)

We consid er here the contributions to mathematics made by ancient Hindus of the early Vedic age. We focus on ancient scripture books namely the five Samhitas known as Rig Veda Samhita, Shukla and Krishna Yajur Veda Samhitas, Sama Veda Samhita and Atharva Veda Samhita, along with their associated BrahmaNa books especially the Shatapatha Brahmana linked to the Shukla Yajur Veda. These are not books on mathematics, but contain considerable mathematical information. The date of all these books is prior to 2900 B.C.E.

The ancient Hindu books directly dealing with Mathematics are the so called Sulba Sutras dated little later than the books mentioned above; there are seven different books. I will mention them only briefly since these books have been discussed elsewhere. My aim is to focus on the books mentioned above.

Among these the Rig Veda Samhita is the oldest dating prior to 3400 B.C.E. It has 1019 hymns and has more than ten thousand verses mainly couplets.

Highlights

1. The error-free methods of chanting the Rig Vedic Samhita closely related to the modern error correcting and detection methods in computers and communication theory.
2. The decimal system for integers, fractions, division and multiplication.
3. The so called Pythagorean triples, right angle triangles with sides in integers, the approximation for pi, the ratio of circumference to diameter and the square root of two.
4. Various geometric problems dealing with rectangles and trapezoid.

Mathematics in Veda Samhitas

As mentioned earlier, the books are the four (or five) Veda Samhitas. Then follow the Brahmanas books and the Sulba Sutra books in chronological order. Somehow many writers jump to the relatively late Sulba Sutras while discussing ancient Indian mathematics. There is considerable mathematical information, both explicit and implicit in the Veda Samhitas. Among them, the earliest is the Rig Veda Samhita.

All the Veda Samhitas are hymns to the various deities. However these hymns praise all forms of knowledge. There was no rigid distinction of the secular and sacred knowledge. The knowledge of mathematics, the knowledge of geometry, especially as related to the construction of houses and cities were all deemed important and worthy of mantras in the hymns. Some of these hymns dealing with the series of integers are recited even today on sacred occasions. Some of the hymns, which deal with cosmology, imply that these poets were very familiar with geometry and the planning needed to construct complex objects.

Consider for example the following verse in RV (Mandala10, Sukta130, Verse 3). The words in Italics are the words in the Sanskrit original. It deals with the creation or formation of the universe.

- Who was the measurer *prama?*
- What was the model *pratima?*
- What were the building materials for things offered *nidanam ajyam?*
- What is the circumference (of this universe) *paridhih?*
- What are the meters or harmonies behind the Universe *chhandah?*
- What is the triangle (yoke) *praugam* [which connects this universe to the source of driving force, the engine]?

All the words in Sanskrit, *prama* etc., are geometrical terms which also occur in the later Sulba Sutras with the indicated meaning. Hence it is safe to assume that these poets were aware of the construction of buildings and other artifacts.

In Atharva Veda (10.2.31) the town of gods called *Ayodhya* is described. It is circular in plan with eight rampart walls and nine doors. Even if the poem is interpreted metaphorically, the use of a metaphor implies that poets had the experience of real things, i.e. a real physical city. A.V. 19.58.4 declares that the town should be made unconquerable using the thing called ayasa. Whether we translate it as strength or as some metal or as iron which is the word's meaning in later times, either way it indicates that the poets were aware of complex planning of these geometrical entities.

The simplistic notion circulated by the indologists of the nineteenth century that these poets were nomads with a relatively low level of culture has absolutely no support from the hymns.

Next consider the Rig Veda Samhita which has more than 10,000 metrical couplets. Each verse has a distinct metre which imposes a structure on the verse like the total number of vowels in it and the number of vowels in each subgroup of the verse. Moreover the medium of preservation of the text was also recitation. The Vedic sages attached the greatest importance to the preservation of the text of the Rig Veda along with accent marks and developed special methods of recitation which remind us of the modern error correction and detection codes in modern communication and computer systems, i.e. what the codes do for correcting the linear printed text is done by these special methods of recitation for oral text.

Chariots are mentioned copiously in Rig Veda (1.102.3, 1.53.9, 1.55.7, 1.141.8, 2.12.8, 4.46.2...). Chariots could also be triangular having three seats and three wheels (RV. 1.118.2, RV 1.34.2).

A spoked wheel is mentioned in many places in Rig Veda. Specifically the five spoked wheel (1.164.13) and the 360-spoked wheel (1.164.48) are mentioned. The spoked wheel has four parts, hubs *nabih*, fellies *pradhaya*, spokes *shanka,* are or rim. By the time of Yajurveda (Y.V. 16.27) the number and varieties of the manufactured chariots had increased so much that a separate guild of chariot makers was developed. Dr. Kulakarni (1983) writes

“The proficiency in chariot building presupposes a good deal of knowledge of geometry... The fixing of spokes of odd or even numbers require knowledge of dividing the area of the circle into the desired numbers of small parts of equal area, by drawing diameters. This also presupposes the knowledge of dividing a given angle into equal parts”.

Finally we come to the role of rituals. The Rig Veda is full of references to the words which come up in rituals, even though it does not mention any ritual in detail. The details of the rituals, especially the design of the fire altars and methods of constructing them using specially shaped bricks are given in the subsequent Brahmana books and also, with more details, in the Sulba Sutra books, the mathematical texts of the late Vedic period. Whenever any religious rituals are codified in a oral or written, the implication is that they must have been in existence for much longer time. For instance consider the three types of fire altars namely Garhapatya, Ahavaniya and Dakshina. All three are mentioned in Rig Veda. However the Shatapatha Brahmana declares that the three fire altars are square, circular and semi circular in shape and, more importantly having the same area. All scholars such as Burk (1901), Seidenberg (1962, 1978), Dutta (1932) agree that this constraint of equal area must have been there even in the early Vedic age before the codification of Rig Vedic Hymns. To construct a figure of a specific area, we need to have at least an approximate method of finding the square root of the number two. To construct a circle of area equal to that of a square, one needs to have at least an approximate value of the number pi, the ratio of circumference of a circle to the diameter. Again the books like the Shatapatha Brahmana or the Sulba Sutras codify the methods existing for a long time in addition to developing new methods of drawing the geometrical figures and the associated theory. The minimum knowledge needed for finding the square root of two or for drawing isosceles trapezium is that of Pythagorean triples like $3^2 + 4^2 = 5^2$. All this evidence implies that the Vedic sages in the early Vedic age knew some simplified versions of the Pythagorean theorem. It was there, probably that the origin of mathematics took place as argued by Seidenberg (1978), Rajaram and Frawley (1995).

Method of chanting based on error correcting codes

Rig Veda is a book of about 10,000 verses composed several thousand of years ago. Still all the available manuscripts and the authorised audio versions recorded all over India differ from each other in only one syllable. Such a feat is possible because Rig Vedic sages had developed methods of chanting reminiscent of the modern methods of error detecting codes, an advanced mathematical concept and associated technology developed only in the last fifty years of this century.

With the ubiquity of the transmission of strings of symbols over wires and wireless, it is easily realised that the string of symbols received by an user over wireless, is quite different from that sent by the sender because the symbol string received by the user has been corrupted by unwanted symbols, labeled noise. The system detects and corrects all the inserted errors; It should be obvious that if only the string of message symbols were transmitted, the received message would be a undecipherable because of the inevitable errors inserted the transmitting medium. Error detection and correction is possible only because, in addition to the symbols constituting the message, additional symbols, the so called redundant symbols, have to be added to the string to be transmitted. The procedure by which a new string is constructed from the given string by adding additional symbols is called a code in the literature on the Mathematical theory of communication. These coding methods were developed only in fifties of this century.

The users of Rig-Veda were confronted with a similar problem and came up with a similar answer six or seven thousand years earlier. The Rig Vedic sages were sure that their poems would be read in much later ages(Mandala 3, Sukta 33, Verse 8). Speech was the medium of their poetry as well as that of their preservation. In Sanskrit language, words with apparently trivial errors of transposition of syllables could have a vastly differing meaning. There is the story of a titan who wanted to ask a boon that there shall be no gods. The Sanskrit phrase is *nih + devatvam = nirdevatvam*. The titan, being careless, made a transposition error and asked for *nidravatvam,* which means to be enveloped by sleep. Each RV verse is written in a particular metre. For instance the most popular metre is Gayatri, each verse having 24 syllables, divided into three parts of eight syllable each, the so called three “feet”. Each foot is a linguistic unit (or sentence). The errors that are to be detected are:

1. Deletion or addition of a syllable into a word.
2. Deletion or insertion of a word into the sentence.
3. Preservation of the order of words.
4. Avoiding long jumps, i.e. Suppose there are two words **a** and **b** which are phonetically close. Let **a** and **b** occur in verses numbered **x** and **y**. Let **c** and **d** be the words next to **a** and **b** in the corresponding verses. There is a tendency while chanting to jump from a word **a** in verse **x** to word **d** in verse **y**; similarly to jump from word **b** in verse **y** to word **c** in verse **x**. This error is serious.

The sages developed several systematic methods of chanting (or codes) so that the errors would not only show up, but also the correction also becomes clear. Hence for each verse there is the standard method of recitation as well as these special methods recitation called here as codes. The code is formed by adding additional words, several times the number in the original. The codes have suggestive name, like krama (succession), mala (garland), jata (the matted hair), danda (stick) and ghana, the hard one, the last one being the most comprehensive one. I will mention here only two such methods or codes named mala (garland) and the double wheeled chariot, the latter deciphered for correcting the error of type 4, the so called long jumps.

MALA (Garland)

The first step is to break up the verse with euphonic combinations into individual words. Consider one half of the verse RV (10.97.22). It has six words divided into 2 parts or feet. Label the words a_1, ..., a_6

Line 1: a_1 a_2 a_3 a_4 a_5 a_6

Rearrange the above line as six lines as shown below

a_1 a_2
a_2 a_3
a_3 a_4
a_4 a_5
a_5 a_6

a_6 a_6

Make a copy of the matrix patter, flip it top to bottom and right to left keep it next to the original as indicated below so that the bottom lines are lined up.

a_1	a_2	a_6	a_6
a_2	a_3	a_6	a_5
a_3	a_4	a_5	a_4
a_4	a_5	a_4	a_3
a_5	a_6	a_3	a_2
a_6	a_6	a_2	a_1

The above diagram with 6 lines looks like a garland, as the name indicates. Now chant the six lines above together as a single verse, one by one beginning with the top, left to right. The verse has 32 words since each is repeated four times.

Call this verse i_1, i_2, ..., i_{32}, each i_k being a distinct word. This verse, Vikrati is repeated below, after bending it in the middle and reversing it

a_1 a_2 a_6 a_6 a_2 a_3 a_6 a_5 a_3 a_4 a_5 a_4
a_1 a_2 a_6 a_6 a_2 a_3 a_6 a_5 a_3 a_4 a_5 a_4

The highest number known to Greek is 10^4

Notice that each of the 16 columns has only one entry.

When the verse of 32 words is recited, the reciter may make unconscious errors; let the verse heard by another person be indicated below, each O_i being a word. Again bend the string in the middle and reverse it

O_1 O_2 O_3 .... O_{16}
O_{32} O_{31} O_{30} .... O_{17}

If the verse recitation were perfect, there would be only one entry in each column, i.e $O_1 = O_{32}$, $O_2 = O_{31}$. If for example O_2 is different from O_{31}, then there is an indication of error.

Now note that every word like a_2 occurs at least four times in the chanted Vikrati verse (*). After imposing appropriate assumptions on the pattern of errors, and assuming no error in the first word a_1, one can prove [11] that the correct verse can be recovered namely:

$a_2 = \text{MAJ}\{O_2, O_{31}, O_5, O_{28}\}$

where MAJ {., ., ., .} means the word among the four which occurs more than others. The details of the mathematical arguments are in [11].

Two wheeled chariot (*Àvichakra ratha*)

This is a code that can handle two verses which end with different words which are phonetically close, i.e. error of type 4.

Verse 1.1.1:	1	2	3	4	5	6	7	8
Verse 1.20.1:	a	b	c	d	e	f	g	h

8: ratnadhatamam g: ratnadhatamah

The chanting procedure ensures that word 8 is chained to 7 and the word g is chained to f and no jump can occur from 7 to g or f to 8.

Arithmetic: Numbers and decimal system

The names for the numbers one to nine found in Rig Veda are eka, dvi, tri, chatur, pancha, shat, sapta, asta, nava. The names for ten, twenty, ….., ninety occur in RV (2.18.5-6). The intermediate numbers have appropriate names. For instance ninety-four is termed four plus ninety. Nineteen is expressed one less than twenty etc. The RV (3.9.9) has a number 3339 spelled as three thousand, three hundred and thirty nine. The RV (2.14.16) uses the word hundred thousand, the modern lakh. Many lakhs are described as hundreds of thousands in RV (1.14.7). Rig Veda has more than a hundred references to numbers.

The Shukla Yajur Veda (17.2) mentions ayuta (10^4) niyuta the series of 10 upto 10^{12} in steps of powers of 10 namely sahasra (10^4) niyuta (10^5), prayuta (10^6) arbuda (10^7), hyabuda (10^8), samudra (10^9), madhya (10^{10}), anta (10^4), parardha (10^{12}) etc. A similar list is in Taithiriya Samhita 4.40.11.4 and 7.2.20.1; Maitrayani Samhita 2.8.14; Kathaka Samhita 17.10 etc.

The atharva veda Samhita (6.25.1 thru 6.25.3, 7.4.1) specially emphasises the common relationship between one and ten, three and thirty, five and fifty, nine and ninety, clearly indicating that these persons had a good grasp of the basics of decimal system for positive integers.

The Yajur Veda (Y.V. 18.24 thru 26) mentions the series of odd numbers 1,3…33 and the series 4,8,….48. The Taittiriya Samhita (7.2.11 through 20) has in addition, the series 10,20…100, 100,200…1000; 10,100,1000…upto 10^{12} and the multiplication

4 x 25=100=5 x 20=10 x 10=20 x 5.

MULTIPLICATION AND DIVISION

Shatapatha Brahmana gives many instances of multiplication . For example (2.3.4.19-20) gives 360 x 2=720, 720 x 80 = 57,600.

Again the same book in the section (10.24.2 1-20) gives the result of dividing 720 by all the integers from 2 to 23 which do not give any residue. For instance it considers 720/2, 720/3, 720/4, 720/5, 720/6, 720/8, 720/9, 720/10, 720/12, 720/15, 720/16, 720/18, 720/20, 720/24.

Fractions:

Rig Veda (10.90) mentions the fractions ¼, ½, ¾ and also the fact ¼ + ¾ = 1. Shatapatha Brahmana mentions these and similar results, In addition it mentions in (4.6.7.3) that 1/3 + 2/3 = 1 Y.V. (18.26) mentions the series 1/2 1 1/2, 2, 2 1/2,3, 3 1/2 and 4.

Zero:

The standard question is was there the knowledge of the arithmetic symbol zero? Even though the Vedic sages know the similarity of the relationship between 1 and 10, 2 and 20 etc, there is no explicit mention of the place value system or of the numeral zero. We want to caution that the Vedic poets used extensive symbolism. whether they expressed the numbers in a code language like the code of Aryabhatta or the code Ka¿apayadi remains to be investigated.

The concept of infinity:

The Vedic Indians were aware of the fundamental difference between a large number and infinity. They were aware that an infinite number couldn't be produced by several finite numbers with finite number of operations.

These are many words for infinity namely ananta, purnam and *aditi*. The word innumerable *asamkhyata* occurs in Y.V. 16.54. Brhad Aranyaka Upanishad (2.5.10) – (the Upanishad associated with Shatapata Brahmana and Shukla Yajur Veda) in describing the count of the mysteries of Indra declares it is *ananta* literally meaning that which has no end *anta*. They stated two clear definitions. The Atharva Veda 10.8.24 states that "infinity can come out of only infinity" and "infinity is left over from infinity after operations on it". These two statements are made more precise in the invocatory verse of Isha Upanishad (chap.40, Shukla Yajur Veda).

From infinity is born infinity.

When infinity is taken out of infinity,
Only infinity is left over.

The purna is not limited to the mathematical infinity. The author of the hymns is trying to define the concept of all ‘perfect’ perfects.’ Its projection to the realm of mathematics is the mathematical infinity denoted by the symbol infinity later.

Shatapatha Bramana text

(Right angled triangles, Pythagorean triples, square root ...)

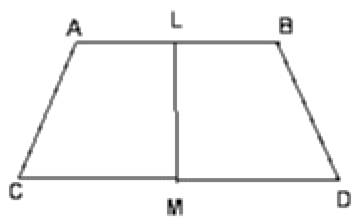
All the problems solved in the Sulba Sutras are mentioned or partially solved in the Shatapatha Brahmana (SB) As Kulkarni (1981) notes:

1. The different types of chits fire altars described in the Sulba Sutras are in SB (6.7.2.8)
2. Two types of uttaravedish fire-altars (out of six) are described in SB (7.31.27)
3. The numbers of bricks need to construct different types of Dhishnya fire altars mentioned in S.3 (9.43. 6-8)

It discusses the problem of scaling i.e. given a square shaped altar of area 7½ square purushas, how to increase its dimensions so that area becomes 101.5 square purusha. The solution is given Kathyayana Sulba Sutras. But the original solution is in SB (10.2.37) and in other different places. However the solutions given in different places of SB are different indicating that the field of algebraic geometry was still developing and was in a state of flux.

A popular question is did the Vedic sages know the so called Pythagorean theorem, This theorem in its simpler version for integers, i.e. the knowledge of triples like (3,4,5) obeying $3^2 + 4^2 = 5^2$ was known in the Smhita period. The answer is a categorically yes. The writers of the Shatapatha Brahmana knew this knowledge. The theorem and its converse were stated precisely by Baudhayana in his Sulba Sutras. We have to realise that the authors of the Sutras incorporated into their books all that was known earlier in addition to their own findings and Baudhayana's Sulba Sutra the earliest known book on mathematics is no exception.

The word akshnaya occurring in several places of the Taittiriya Samhita, Krishna Yajur Veda, 5.2.10, 5.2.7, 5.3.5, 6.2.8, 6.3.10 etc., is the hypotenuse of a right angled triangle or the diagonal of a square or rectangle or trapezium. Shatapatha Brahmana (3.5.1 to 6) gives a method for constructing a isosceles trapezium shaped fire-altar with parallel sides being 24 and 30 and the height or distance between the parallel sides being 36. Of course, the description is given in linear prose without resort to figures.



AL=LB=12, CM=MD=15, LM=36

How did they ensure that the sides AB and CD are parallel? There are only 2 possibilities: either they had access to a device for drawing right angle triangles, for which there is no evidence. Or they knew the side CB=AD=45 using the knowledge of the Pythagorean triple $3^2 + 4^2 = 5^2$

$$CB^2 = 36^2 + (12+12+3)^2 = 36^2 + 27^2 = 9^2\,(4^2 + 3^2) = 9^2\,5^2 = (45)^2$$

So points B and A were located so that CB=AD=45 and located the midpoint L. Either way, the knowledge of the triple for integers was and its relation to right angle triangle must have been known.

There is another evidence to support the idea that the authors of Shatapatha Brahmana knew the Pythagorean theorem for integers. As Dutta[5] has noted, the verse 13.8.1.5 suggests that a particular type of altar named *paitrki vedi* whose corners point to the four directions must be *half* the area of the regular square vedi whose four sides point to the four directions. Clearly the Shatapatha Brahmana must have known about the solution to the problem they noted. The *Paitrki vedi* square is constructed from the regular vedi squares by joining the mid points of adjacent sides. The fact that this altar has half the area of the regular square is a clear indication of their knowledge of the Pythagorean triple. The construction of the Paitrki vedi from the regular vedi is indicated in Baudhayana Sulba Sutras 3.11.

Mathenmatics of Sulba Sutras

We give only a brief overview of the four Sulba Sutra books associated with the names of Baudhayana, Apastamba, Katyayana and Manava. The word *sulva* is derived for the root *sulv*, to measure. Since the cord or rope, rajju was used for measuring, in course of time *sulva* became a synonym for rope. The date of composition of the earliest these books must be much before 1800 B.C.E when the Sarasvati river dried up and the Vedic civilisation was on the decline.

The following geometrical theorems are either explicitly mentioned or clearly implied in the construction of the altars of the prescribed shapes and sizes [Ramachandra Rao, 1997]

1. The diagonals of a rectangle divide the rectangle in four parts, two and two (Vertical, opposite) which are identical (Ban (iii, 168,169,178)
2. Diagonals of a rhombus bisect each other at right angles
3. An isosceles triangle is divided into two identical halves by the line joining up the vertex to the middle point of base (Bau, iii, 256)
4. A quadrilateral formed by the lines joining middle points of a rectangle is a rhombus whose area is half of that of the rectangle.
5. A parallelogram and a rectangle on the same base and within the same parallels have the same area

Baudhayana theorem (earlier to 2000 B.C.E) (called as pythogoras theorem)

The diagonal of a rectangle produces both areas which the length and breadth produce separately This theorem is usually attributed to the Pythogoras (6^{th} B.C.E)

The Baudhayana work even states its converse.

Value of pi (ratio of circumference to diameter)

There are eight different approximations to pi in the different Sulba Sutra, Baudhayana (1.61) gives the best approximation among them namely 3.088. The closest value to modern value is given by later Sutra work Manava Sulba Sutra (1.27) namely:

$4/ (1\ 1/8)^2 = 3.16049$

Square root of 2 and other surds

The earlier known value of the square root of two is obtained from one of the cuneiform tablets for the Babylonian times (1600 B.C.E) [Neugebeauer, 1952], given in sexagesimal notation Baudhayana gives the following approximation

Square root of 2 =1 +1/3 + 1/(3 x 4) –1/(3 x 4 x 34) This is a better approximation than the earlier one. For details see R.P. Kulkarni's book(1983)

Mathematics in the Indus Seals

In 1875, in the archeological excavation near the town of Montgomery (now in Pakistan), more than three thousand inscriptions were found, all in an undeciphered script. There have been many attempts to decipher the script, with very little success. Still because of the prejudices of the early indologists (especially their Indian followers), we find frequently the claim that these indus seals have no connection with the Rig Vedic texts. If these seals have not been deciphered, how can one make such definitive statements. As Sri Aurobindo points out, in early indological literature, a mere conjecture after repeated cross references by different scholars acquires the status of "truth". All the seals are dated 2000-3000 B.C.E by radio carbon dating methods.

Recently N.K. Jha has discovered definitive clues in deciphering the script. It is related to the old Brahmi, not the Ashokan Brahmi. It has only three symbols for vowels and 31 symbols for the consonants of the Sanskrit. Different symbols are compounded as in Sanskrit writing. It is written both for left to right and right to left. Jha has shown that 100 of the seals contain the one hundred words of the glossory on the Vedas developed by the earliest known lexicographer of India, yaska. The deciphering technique is fairly definitive.

The seals give the symbols for the numerals one through nine, ten, twenty, thirty, hundred, thousand and hundred thousand. The symbols for one through nine are close to the corresponding Roman symbols, except that alternatives are there for many of them.

Next there is one seal having three characters which have been deciphered as p k 10. k has been known to be the symbol for karni, the square root, in many places including the Bakshali manuscript of 200 AD. According to Jha, p is the abbreviated form for 'paridhi vyas anupat' (the circumference diameter ratio). Thus the above seal gives the square root of ten as an approximation for the pi. This approximation is 3.16. This statement also offers a solution to the long-standing problem. Why was the ratio named pi in Greek. This approximation for pi has been mentioned in the later Jain mathematical literature (500 B.C.E)

Conclusion

We have given in some detail of the mathematics found in the Rig Veda and other Veda Samhitas, and the Shatapatha Brahmana and other Brahmana books. These books indicate the Indians of the early Vedic age knew the decimal systems for integers including multiplication and division, the problem of increasing the square of a certain area to another of predetermined area, problems associated with isosceles trapezium. It is evident that these early Indians knew the so-called Pythagorean triples for integers, approximation to the square root 2 and 3, approximate to pi etc.

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Error Correcting code

Abstract

Error correcting code-like chanting procedures in ancient India.

by **R. L. Kashyap** and **M. R. Bell**

The integrity of the oral version of the Rig Veda Samhita, the ancient sacred book of the Hindus, dated at least earlier to 2000 BCE, has been preserved by the use of special chanting procedures called *vikrati*s which resemble the modern error correcting codes developed in the last fifty years for error free transmission of messages or strings of symbols over electronic media. This book has more than ten thousand verses in a variety of metres. We consider here one particular error correcting code-like chanting called as Krama-*maala Vikrati* and show its connection to the modern Linear Block Code developed on the basis of advanced mathematical concepts. We also mention the other error detecting procedures (not correcting) for Rig Veda Samhita based on the use of binary numbers dated 200 BCE or earlier.

Introduction and Summary

The theory and technology of error correcting codes have been developed in the last fifty years for error free transmission of messages or strings of symbols over electronic media which introduces disturbances or errors into the message. Thus it is of interest of note that several millennia ago error correcting code-like procedures were developed for chanting the Rig Veda Samhita, the Sacred Sanskrit text of the Hindus so that the complete text even after oral transmission over several millennia has not been corrupted at all except for one syllable in a single verse out of about ten thousand verses [1].

The key idea behind the special chanting methods called as *Vikrati*s and modern error correcting codes is same namely the use of encoder and decoder. The modern error correcting methods are based on the advanced mathematical concepts such as group theory. The sages seem to have arrived at similar results purely by intuition. Specifically consider the correct preservation of the chanting of one verse M consisting of m words. The *Vikrati* will generate another verse M' based on the words of M having many more words, typically 4 m or more.

Let the verse O be the corrupted version of M', the version recited by a chanter who makes unconscious errors. Applying mentally the decoder idea - reverse of the encoder idea - to the corrupted output of O yields the correct verse M under appropriate conditions. Alternatively another person who is familiar with the decoder and hears the output O can suggest the correction. The sages were very much aware of the trade off between the redundancy in the code quantified by the length of M' versus the error.

Since there are many different types of errors, the sages realized that only one encoder or *vikrati* is not enough. There are eight families of *Vikrati*s each having many members. Using all these methods in totality results in the almost zero error feature quoted earlier. We focus in this paper on one particular procedure namely Krama-*maala Vikrati*.

We also mention the error detection schemes - not error correcting - applicable to ordinary chanting of the verses without involving the elaborate *vikrati*.

The appendix has the original description of the two *vikrati*s in ancient texts in Sanskrit with English transliteration and their translations.

Text of Rig Veda (RV)

Rig Veda Samhita happens not only to be the most ancient poetry in humankind, but also the principal religious composition of the Hindus. It consists of 1017 hymns with a total of about ten thousand verses, mainly couplets, divided into ten mandalas (circles) or cantoes. The language of these hymns is Vedic Sanskrit. The entire work is metrical, each verse being in one of fifteen metres. The composition is dated prior to 3400 B.C.E. The composition was preserved by oral transmission, the teacher training his students in these chants. The oldest manuscript of the Rig Veda Samhita is dated circa 500 CE. Hence the methods used for preserving the integrity of the contents are based on the recitation methods only, not on the manuscript.

Hinduism was and is highly decentralized in its structure without any central organization charged with preserving the integrity of their composition. Still all the manuscripts available and the chanting of experts from different parts of India are almost identical to one another. There are exactly two different versions differing from each one other in one syllable of one particular verse among the ten thousand verses [1]. Such an achievement, achieved no where else in the world (or even in India regarding their other books) is possible only because special chanting methods were developed so that the knowledge of the rules behind these chanting procedures allowed the reciter to detect the errors in his own chanting or the chanting of others.

Every syllable of every verse has to be intoned in one of three ways. The written text has intonation marks. These intonations have to be preserved also. The intonation is based on a group of context sensitive rules. Preservation of the integrity of these intonations is one of the key features of the error correcting procedures. We will not discuss this topic here.

Most Rig Vedic verses are couplets, each line of couplet having a small number of words numbering say from three to about 10 or 15. The text is available in two versions. In the first version, *Pada* version, every word in the verse is clearly delineated and it is meant for study and error correction. The second version is the Samhita version appropriate for music-like chanting where neighbouring words are combined by certain rules of euphony so that the entire verse can be chanted without any breaks. For instance two distinct neighbouring words in the *pada* version like *rama* and *isha* such that ending vowel of first word is *a* and the beginning vowel of the second is *i*, then the two words together are chanted as *ramesha,* i.e. *a* and *i* together becomes *e*.

The *Pada* version is recited without any break between words and without altering the identity of the words.

ERRORS

Error free recitation implies the correct recitation of every word in the appropriate order with the appropriate intonation on every syllable. In this paper we focus only on the integrity of text of word and the order of words in a hymn.

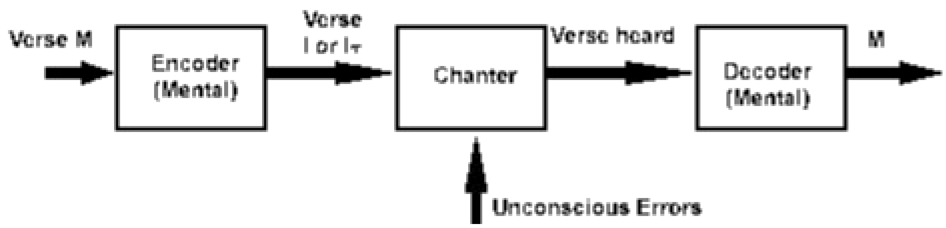
Even though the reciters are careful by training, still one word x in a verse is recited unconsciously as x', where x' is a word phonetically close to x occurring in some other verse. For instance in RV there are many occurrences of words ending with pati, but only a small number ending with pata, leading to the type of error mentioned above.

Even though the order of words in a Sanskrit sentence is not relevant as far as meaning is concerned, still the preservation of the word order is crucial for preserving the sound quality. So there are several methods of binding pairs of adjacent words in a verse. Still the transposition error i.e. flipping the order of a pair of adjacent words has to be taken care of.

There are also several special error families. For instance a common error is jumping from one word in a verse to completely another word in a completely different verse because of the similarity of the context. We will mention ways of handling some of these error.

Vikratis or Encoder

Consider a RV verse labeled M having m words, M = $a_1, a_2, a_3, \ldots, a_m$, each a_i being a distinct word. Note two consecutive words a_i and a_{i+1} are always different from each other. The idea of error correcting code-like *vikrati*s is given below. It has two boxes, the encoder and decoder. It should be noted that both encoder operations and decoder operations are carried out mentally without using mechanical gadgets. We use these words because of their familiarity in modern parlance.



For the given verse M, the encoder generates a verse I having n words, n being much greater than m. The words of I are those in M. The chanter recites I unconsciously, introducing errors. The output heard by a listener is, O = $O_1, \ldots, O_n$, having n words, n being known. Thus, O # I.

(# - Not equal to)

With every M, a string I is generated , having the same words as in M; the length of I being 4m or more. The rules for generating the verse I should be such that I can be chanted with relative ease. The verse I is chanted by a reciter. Typically, he or she memorises all the encoded versions of all the verses in one mandala or chapter of Rig Veda. Hence errors are bound to creep in. The version of I heard is called O, which may have errors. Another expert in the same family, on hearing O, acts as a (mental) decoder and generates the correct verse M, under appropriate conditions.

There are eight families of *vikrati*s with suggestive names like *maala* (garland), *ratha* (chariot), *danda* (stick), *dvaja* (flag). Each family has several variants. We give below a variant of the garland or *maala* family, *Krama-maala*.

Krama Mala (progressive – garland)

The definition of *Krama-maala* is described in a phrase [1] detailed in the appendix. Consider a verse $M = a_1,...,a_m, a_m$ having m words a_i, $i = 1,...,m$. Krama means making m pairs of adjacent words and arranging them as a matrix. Note the repetition in the m^{th} line.

$a_1 \quad a_2$

$a_2 \quad a_3$

......

$a_{m-1} \quad a_m$

$a_m \quad a_m$

Next take a copy, flip it left to right , top to bottom and place it next to the original matrix so that the bottom lines of the two matrices are in line, giving the impression of a garland.

$a_1 \quad a_2 \quad a_m \quad a_m$

$a_2 \quad a_{-3} \quad a_m \quad a_{m-1}$

....

$a_{m-1} \; a_m \quad a_3 \quad a_2$

$a_m \quad a_m \quad a_2 \quad a_1$

String the m lines together into a single line, left to right, top to bottom.

$I = a_1,......,a_m \quad a_2,......a_{m-1} \quad a_m a_m a_2 a_1$

$= i_1 \, i_2 \, i_3 \, ...i_{4m-1} \, i_{4m}$

This is the output of encoder, labelled as I, and the verse to be chanted, having 4m words. In I, right half is mirror-symmetrical of the left. Hence, bend I in the middle and place the right half below the left half, to look like a matrix with 2 rows and 2m columns.

$$I = \begin{bmatrix} i_1 & i_2 & .... & i_{2m} \\ i_{4m} & i_{4m-1} & ... & i_{2m+1} \end{bmatrix}$$

It should not be a surprise that every column in I has only one distinct member among $a_1,...,a_m$.

We assume that i1 will not be corrupted. Using I, we can recover M in 4 ways, namely $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, $M^{(4)}$.

$M^{(1)} = \{ i_1, i_5, i_9, ... \}$

$= \{ i_{4(k-1)+1}, k=1,..,n \}$

$M^{(2)} = \{ a_1, i_2, i_6, i_{10},... \}$

$= \{ a_1, i_{4(k-1)+2}, k=1,...,(m-1) \}$

$M^{(3)}$ is the mirror reflection of $M^{(2)}$ and $M^{(4)}$, that of $M^{(1)}$.

$M^{(4)} = \{ i_{4m}, i_{4m-4},... \}$

$= \{ i_{4(m-k)}, k=0,1,...,(m-1) \}$

$M^{(3)} = \{ a_1, i_{4(m-k)}, k=0,1,...,(m-1) \}$

This suggests the decoding procedure. When I is recited, let the output heard be O, which differs from I because of the unconscious errors introduced by the reciter. Some of the words of O may not be in I. Assume that O has 4m words and that there is no error in the first symbol of I. Write the output string as 2 x 2m matrix, by bending O at midpoint and twisting it to the left.

$$O = \begin{bmatrix} o_1 & o_2 & ..... & o_{2m} \\ o_{4m} & o_{4m-1} & ... & o_{2m+1} \end{bmatrix}$$

All the columns in O having 2 distinct words have an error. We do not know which is the error. Based on the M_i, i = 1,...,4, write four estimates of M, from O, labeled $M'_1,...,M'_4$.

$M'_1 = \{ O_1, O_5, O_9,... \}$

$= \{ O_{4(k-1)+1}, k=1,...,m \}$

$M'_2 = \{ a_1, i_2, i_6 ... \}$

$= \{a_1, O_{4(k-1)+2}, k=1,...,(m-1) \}$

$M'_4 = \{O_{4(m-k)}, k=0,1,...,(m-1) \}$

$M'_3 = \{a_1, O_{4(m-k)}, k=0,\ldots,(m-1)\}$

Merge the four estimates by a majority vote on the corresponding members.

$M' = \{ a_1, MAJ[O_5, O_2, O_{4m-4}, O_{4m-1}], MAJ[O_9, O_3, \ldots], \ldots \}$

$MAJ[x_i, i = 1,\ldots,4]$ = majority vote operation, the symbol occurring most among x_i.

Theorem:

1. Assume that there is no error in a1,
2. Assume that there are no more than two errors per ai among its four appearances,
3. No error is repeated, i.e. if a2 is modified into a'2 and a''2 in M'i and M'j, then a'2 # a''2.

Under these conditions, output of the decoder equals the correct value.

Comment I : The system thus allows for for at most 2(m-1) errors among the 4m words in I, no error if the same symbol being repeated.

Comment II : The only information being used is the output and the rules behind the encoder.

Example 1: Let m = 4. Let M = a b c d.
Construction of encoder:

a b d d
b c d c
c d c b
d d b a

I = a b d d b c d c
a b d d b c d c

Let O = a b' d' d b c' d c @ errors = 6
a b d d'' b'' c d c''

where b' # b'', etc.

Then the four estimates M'_i are:

M'_1 = a b c'' d''
M'_2 = a b' c' d
M'_3 = a b c d
M'_4 = a b'' c d

U M'_i = a b c d = M

MAJ

i= 1,…,4

Details:

From the half verse M, we construct M' using the encoder rules or *vikrati*. Again, M' must be in a form which can be recited. Reciting the same verse many times does not guarantee any error correcting capability because boredom in the repetition increases the frequency of occurrence of errors.

Hierarchy of reciters :

Typically, in each town or region, one group memorizes one particular Mandala (or Canto) of Rig Veda, having about 1000 verses. The members use standard Pada and Samhita versions. Then, for each mandala, there are persons who recite the Vikrati form of all these verses, with one particular Vikrati. The output of each reciter of a particular Vikrati is tested by both herself/himself and her/his fellow experts in that Vikrati. The mental load is not considered heavy. Recall that each Vikrati output is deemed to produce the correct output under certain conditions. Then the persons who recite different Vikratis pool together their results and get the correct result without any qualifications. The existence of such assemblies is mentioned in the last hymn of Rig Veda namely Mandala 10, sukta 191.

Error Correcting Method

There are also procedures which indicate that an error is there in the chanting, but cannot correct it. These procedures are far simpler and do not involve the elaborate recitation of the *vikrati* form. The idea is described in the book "Chandas - Shastra" circa 200 CE or earlier - ("The Science of poetic metres"). Every syllable in a verse is either heavy or light in pronunciation. Thus one can associate a binary string for a verse, the numeral one indicating the heavy syllable and zero or some symbol indicating the light one. This binary string is converted into a decimal number by the usual method namely multiply the kth digit in the string by 2 raised to the power of k and add all of them. This number acts like a signature of the verse. The book mentions recovering the binary string from the decimal number and vice versa. The person responsible for that portion of chant of RV memorizes the first phrase of each verse along with its decimal number. Periodically the reciter computes this number from his/her current pronunciation and checks whether it equals its ideal number. If it does not match, it indicates error. Even though the book mentioned has been translated into English about a hundred years ago, only recently did the deep meaning in it was noticed by Professor Van Nooten [3].

Discussion

One may wonder whether the need for these procedures exists since the manuscripts or books of Rig Veda Samhita are easily available, the latest available manuscript being dated prior to 500 CE. The answer is yes. We have focussed only on the syllables in the text. But every syllable has to be intoned in a particular way and the intonation cannot be completely specified by written symbols. The correct preservation of the oral version with proper intonations still needs the *vikrati*s. The chanting of the forms are treasured regardless of their utility as an error-correcting code.

Rig Veda Samhita has been the object of intense study in western universities for more than one hundred and fifty years, the interest tapering off only in the last fifty years. The *vikrati*s were often regarded as another instance of the Hindus superstition or obsession to make things more complicated, and thus ignored. Again the conversion of the binary string to a decimal number and vice versa mentioned by Pingala was ignored by the translators like Weber more than 100 years ago. Fritz Staal mentions that the concept of the context sensitive rule in linguistics already mentioned extensively by the Sanskrit grammarian Panini, Circa 500 CE, was not recognized by the so called major experts of the Sanskrit Grammar like Kielhorn or Whitney. Only after the discovery in the West in the twentieth century by Chomsky did Staal and others point out their use almost twentyfive hundred years earlier. What all of these examples state is that the predilection of the earlier western investigators that they knew everything prevented them from understanding the Sanskrit texts of which they claimed to be experts.

Vikratis & modern linear block code

From the previous example as well as the general formulation of the Krama-*maala vikrati*, it is clear that the Krama-*maala vikrati* encoding procedure bears a strong resemblance to that of an algebraic block code. The fact that the message symbols $a_1, \ldots, a_m$ and the resulting codeword symbols $i_1, \ldots, i_m$ come from the same "alphabet" of words suggests a straightforward way of representing the encoding as a rate 1/4 linear block code over a finite field. We will now formulate the example in this framework.

A *finite field* is a finite set of *q* elements on which we can define "addition", "subtraction", "multiplication", and "division". More formally, it is a set *F* of *q* elements with the following properties:

1. *F* is an Abelian group under "+" (the addition operation) with additive identity element 0.
2. *F* – {0} is an Abelian group under "." (the multiplication operation) with multiplicative identity element 1.
3. The distributive law a*(b+c) = a*b + a*c holds for all a,b,c $\in$ *F*.

A finite field with *q* elements is often called a Galois field of order q, and is denoted by *GF(q)*. Galois fields have many distinctive properties [5, 6], but for our purposes, we will only need to know a few key facts:

1. The order *q* of *GF(q)* must be a power of a prime, i.e., $q = p^k$, where p is a prime number, and k = 1,2,...
2. For the case where the order $q = p$, the resulting Galois field *GF(p)* is isometric to the set {0, 1, ... , *p* - 1} with addition and multiplication being given by their arithmetic counterparts mod *p*.
3. *GF(q)* contains an additive inverse -1 corresponding to the multiplicative identity 1.

A *linear block code* **C** is a collection of *n*-tuples from *GF(q)* forming a subspace of the vector space $GF(q)^n$ of *n*-tuples over *GF(q)*. If the dimension of the subspace is *k*, we call the ratio k/n the *rate* of the code [7, 8]. Because $k < n$ (and in fact $k < n$ if the codewords contain any redundancy making them useful for error correction), we can generate the subspace of codewords in the linear block code using the linear relation

$$\mathbf{c} = \mathbf{m}\mathbf{G},$$

where the codewords **c** are of the form

$$\mathbf{c} = (c_1, \ldots, c_n) \in GF(q)^n,$$

the corresponding message **m** to be encoded is

$$\mathbf{m} = (m_1, \ldots, m_k) \in GF(q)^k,$$

and the generator matrix **G** is of the form

$$G = \begin{matrix} g_1 & g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_2 & g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ & & & \cdots & \\ g_k & g_{k,1} & g_{k,2} & \cdots & g_{k,n} \end{matrix}$$

Where the row vectors **g**j = ($g_{j,1}$ $g_{j,2}$, ...,$g_{j,n}$) are basis vectors for the code subspace C. Hence it follows that we can generate all gk codewords in the subspace using the relation

$$\mathbf{mG} = (m_1,\ldots,m_k) \begin{matrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ . \\ . \\ \mathbf{g}_k \end{matrix} = m_1\mathbf{g}_1+m_2\mathbf{g}_2+\ldots+m_k\mathbf{g}_k$$

for all q^k message vectors **m**.

We can now put the *Krama-maala vikrati* example having input message **m** = (a, b, c, d) and codeword c = (a, b, d, d, b, c, d, c, c, d, c, b, d, d, b, a), in the form of a linear block code by assuming the input symbols m_i as coming from a q-ary alphabet tat can be represented by a Galois field $GF(q)$[1]. We then see that generator matrix is of the form

$$\mathbf{G} = \begin{matrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0 \\ 0&0&0&0&0&1&0&1&1&0&1&0&0&0&0&0 \\ 0&0&1&1&0&0&1&0&0&1&0&0&1&1&0&0 \end{matrix}$$

In digital systems in which errors in all codeword symbols are equally likely and the probability that a codeword symbol is in error is less than the probability that the symbol is correct, maximum likelihood decoding is usually accomplished using syndrome decoding [7, 8]. In syndrome *decoding*, we compute the *syndrome*

$$\mathbf{s} = \mathbf{rH} = (r_1,\ldots,r_n)\begin{pmatrix}\mathbf{h}_1 \\ \mathbf{h}_2 \\ . \\ . \\ . \\ \mathbf{h}_{n-k}\end{pmatrix}$$

of the received codeword r = $(r_1,\ldots,r_n)$ (the possibly corrupted version of the true codeword **c)**. Here **H** is the parity check matrix of the code **C**, made up of the row vectors $h_1, \ldots, h_{n-k}$ which are basis vector for the n-k dimensional dual subspace **C_** of the code subspace **C**.

The key observation in syndrome decoding is that if **c** is a valid codeword, then **cH = 0**. Now suppose that the received codeword (possibly containing errors) is given by

$$\mathbf{r} = \mathbf{c} + \mathbf{e},$$

where **c** is the codeword corresponding to the message m and e is the error pattern that when added to the codeword **c** yields the corrupted codeword **r**. Then by linearity, we have

$$\mathbf{s} = \mathbf{r\,H} = (\mathbf{c} + \mathbf{e})\mathbf{H} = \mathbf{cH} + \mathbf{eH} = \mathbf{0} + \mathbf{eH} = \mathbf{eH}.$$

So as long as the error pattern **e** itself is not itself a codeword, **s** will be nonzero, and we will be able to detect the presence of errors. The maximum likelihood estimate of the error pattern e is then simply the minimum weight[2] error pattern $\hat{\mathbf{e}}(\mathbf{s})$ corresponding to the observed **s**. By adding $-\hat{\mathbf{e}}(\mathbf{s})$ to the received codeword **r**, we get the maximum likelihood estimate of the original codeword **c**:

$$\hat{\mathbf{c}} = \mathbf{r} - \hat{\mathbf{e}}(\mathbf{s})$$

and hence the corresponding estimate $\hat{\mathbf{m}}$ of the original message **m**. However, as seen in the example, the decoding done my an individual decoding the chant does not fit this decoding framework. To start with, not all codeword symbols are assumed to have equal probability of containing an error, it is in fact assumed that the first symbol is never in error. Furthermore, when an individual decodes the chant, he often knows an error has occurred before getting to the end of a codeword. This seems to imply that there is some kind of sequential decoding going on. Still, it can be seen that the encoding process fits well into the framework of linear block coding, so in the chanting of the Rig Veda Samita, we see what is perhaps the oldest example of error control codes having a structure very similar to that of linear block codes.

[1] This means that the alphabet size of possible words in the chant must be a power of a prime. If it is not, we can augment the alphabet with additional letters so that it is. While it may seem strange to represent an alphabet of words as a Galois field, since "addition" and "multiplications" of chant words is probably a meaningless concept, we do so anyway in order to represent the Krama-*maala vikrati* as a linear block code.

[2] The weight of a vector or codeword is equal to the number of nonzero elements it contains.

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Appendix

The eight *Vikrati* families are *jata* (matted hair), *maala* (garland), *shikha* (hair tied in a knot), *rekha* (line), *dwajah* (flag), *danda* (rod), *ratha* (chariot) and *ghana* (hard or difficult).

The *maala* has 2 forms, *pushpa-maala* (flower garland) and *Krama-maala* (systematic garland). We focus on the *Krama-maala*. We give the Sanskrit description [1] in a brief, but precise form, the so called *sutra*.

bruyat-krama – viparyasav-ardharcha-asya-adito-antatah | *(line 1)*

antam-chadim-nayedevam-kramamaleti-giyate || *(line 2)*

Word to word meaning:

Line 1: declared-krama-reversed or flipped-one half of a rik or verse-its-beginning-end

Line 2: end-and beginning-knit-*Krama-maala*-sing

Translation:

1. Take one half a verse
2. Arrange it in *Krama* form
3. Flip up right to left and top to bottom
4. Arrange the two[arrays] together with end of one set and the beginning of another to be tied together in the same line.
5. This is *Krama-maala*.

This algorithm was followed in section IV.