Square Roots in the Sulbasutra

Dedicated to Sri Chandrasekharendra Sarasvati

who died in his hundredth year while this paper was being written.

by

David W. Henderson[1]

Department of Mathematics, Cornell University

Ithaca, NY, 14853-7901, USA (e-mail: dwh2@cornell.edu)

In this paper I will present a method for finding the numerical value of square roots that was inspired by the *Sulbasutra* which are Sanskrit texts written by the Vedic Hindu scholars before 600 B.C.. This method works for many numbers and will produce values to any desired degree of accuracy and is more efficient (in the sense of requiring less calculations for the same accuracy) than the divide-and-average method commonly taught today.

Several Sanskrit texts collectively called the *Sulbasutra* were written by the Vedic Hindus starting before 600 B.C. and are thought[2] to be compilations of oral wisdom which may go back to 2000 B.C. These texts have prescriptions for building fire altars, or *Agni*. However, contained in the *Sulbasutra* are sections which constitute a geometry textbook detailing the geometry necessary for designing and constructing the altars. As far as I have been able to determine these are the oldest geometry (or even mathematics) textbooks in existence. It is apparently the oldest applied geometry text.

It was known in the *Sulbasutra* (for example, Sutra 52 of Baudhayana's *Sulbasutram*) that the diagonal of a square is the side of another square with two times the area of the first square as we can see in Figure 1.

Thus, if we consider the side of the original square to be one unit, then the diagonal is the side (or root) of a square of area two, or simply the square root of 2, that is $\sqrt{2}$. The Sanskrit word for this length is *dvi-karani* or, literally, "that which produces 2".

The *Sulbasutra*[3] contain the following prescription for finding the length of the diagonal of a square:

Increase the length [of the side] by its third and this third by its own fourth less the thirty-fourth part of that fourth. The increased length is a small amount in excess (*savi´e¸a*)[4].

Thus the above passage from the *Sulbasutram* gives the approximation:

$$\sqrt{2} \approx 1 + \tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}.$$

I use $\approx$ instead of $=$ indicating that the Vedic Hindus were aware that the length they prescribed is a little too long (*savi´e¸a*). In fact my calculator gives:

$$\sqrt{2} \approx 1.4142135\cdots$$

and the *Sulbasutram*'s value expressed in decimals is

$$\sqrt{2} \approx 1.4142156\cdots$$

So the question arises — how did the Vedic Hindus obtain such an accurate numerical value?

Unfortunately, there is nothing that survives which records how they arrived at this *savi´e¸a*.

There have been several speculations[5] as to how this value was obtained, but no one as far as I can determine has noticed that there is a step-by-step method (based on geometric techniques in the *Sulbasutram*) that will not only obtain the approximation:

$$\sqrt{2} \approx 1 + \tfrac{1}{3} + \tfrac{1}{4} \cdot \tfrac{1}{3} - \tfrac{1}{34} \cdot \tfrac{1}{4} \cdot \tfrac{1}{3},$$

but can also be continued indefinitely to obtain as accurate an approximation as one wishes.

This method will in one more step obtain:

$$\sqrt{2} \approx 1 + \tfrac{1}{3} + \tfrac{1}{4} \cdot \tfrac{1}{3} - \tfrac{1}{34} \cdot \tfrac{1}{4} \cdot \tfrac{1}{3} - \tfrac{1}{1154} \cdot \tfrac{1}{34} \cdot \tfrac{1}{4} \cdot \tfrac{1}{3},$$

where the only numerical computation needed is $1154 = 2[(34)(17) - 1]$ and, moreover, the method shows that the square of this approximation is less than 2 by exactly

$$\frac{1}{(1154 \cdot 34 \cdot 4 \cdot 3)^2} = \frac{1}{221{,}682{,}772{,}224}.$$

The interested reader can check that this approximation is accurate to eleven decimal places.

The object of the remainder of this paper is a discussion of this method and related topics from the Sulbasutram.

Bricks and Units of Length.

In the *Sulbasutram* the *agni* are described as being constructed of bricks of various sizes. Mentioned often are square bricks of side 1 *pradesa* (span of a hand, about 9 inches) on a side. Each *pradesa* was equal to 12 *angula* (finger width, about 3/4 inch) and one *angula* was equal to 34 sesame seeds laid together with their broadest faces touching[6]. Thus the diagonal of a *pradesa* brick had length:

1 *pradesa* + 4 *angula* + 1 *angula* - 1 sesame thickness.

I do not believe it is purely by chance that these units come out this nicely. Notice that this length is too large by roughly one-thousandth of the thickness of a sesame seed. Presumably there was no need for more accuracy in the building of altars!

Dissecting Rectangles and $A^2 + B^2 = C^2$

None of the surviving *Sulbasutra* tell how they found the *savi´e¸a*. However, in Baudhayana's *Sulbasutram* the description of the *savi´e¸a* is the content of Sutras 61-62 and in Sutra 52 he gives the constructions depicted in Figure 1. Moreover in Sutra 54 he gives a method for constructing geometrically the square which has the same area as any given rectangle. If N is any number then a rectangle of sides N and 1 has the same area as a square with side equal to the square root of N. Thus Sutra 54 give a construction of the square root of N as a length. So let us see if this hints at a method for finding numerical approximations of square roots. The first step of Baudhayana's geometric process is:

If you wish to turn a rectangle into a square, take the shorter side of the rectangle for the side of a square, divide the remainder into two parts and, inverting, join those two parts to two sides of the square.

See the Figure 2. This process changes the rectangle into a figure with the same area which is a large square with a small square cut out of its corner.

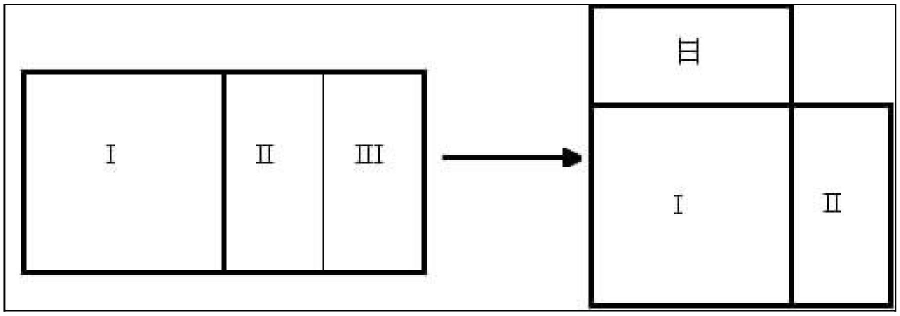

Figure 2

In Sutra 51 Baudhayana had previously shown how to construct a square which has the same area as the difference of two squares. In addition, Sutra 50 describes how to construct a square which is equal to the sum of two squares. Sutras 50, 51 and 52 are related directly to Sutra 48 which states:

The diagonal of a rectangle produces by itself both the areas which the two sides of the rectangle produce separately.

This Sutra 48 is a clear statement of what was later to be called the "Pythagorean Theorem" (Pythagoras lived about 500 BC). In addition, Baudhayana lists the following examples of integral sides and diagonal for rectangles (what we now call "Pythagorean Triples"):

(3,4,5), (5,12,13), (7,24,25), (8,15,17), (9,12,15), (12,35,37), (15,36,39)

which the *Sulbasutram* used in its various methods for constructing right angles.

Construction of the Savi´e¸a for the Square Root of Two

If we apply Sutra 54 to the union of two squares each with sides of 1 *pradesa* we get a square with side 1½ *pradesa* from which a square of side ½ *pradesa* had been removed. See the Figure 2.

Now we can attempt to take a strip from the left and bottom of the large square — the strips are to be just thin enough that they will fill in the little removed square. The pieces filling in the little square will have length 1/2 and six of these lengths will fit along the bottom and left of the large square. The reader can then see that strips of thickness (1/6)(1/2) *pradesa* (= 1 *angula*) will (almost) work:

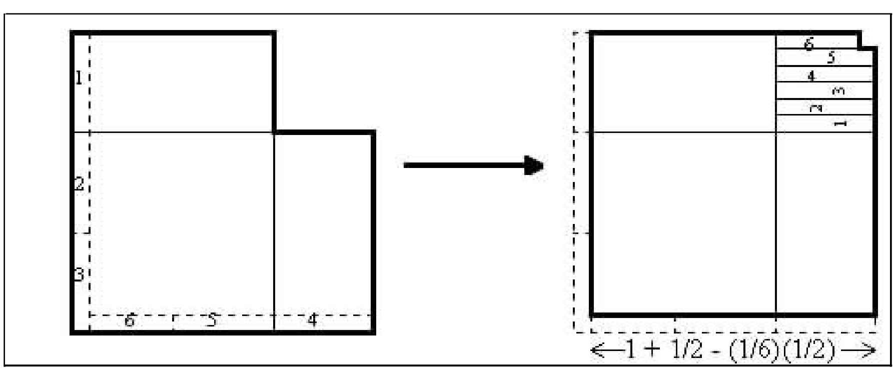

Figure 3

There is still a little square left out of the upper right corner because the thin strips overlapped in the lower left corner. Notice that

$$\left(1+\tfrac{1}{2}-\tfrac{1}{6}\cdot\tfrac{1}{2}\right)\textit{pradesa} = 1\,\textit{pradesa} + 5\,\textit{angulas} = \left(1+\tfrac{1}{3}+\tfrac{1}{4}\cdot\tfrac{1}{3}\right)\textit{pradesa}.$$

We can get directly to $1+\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{3}$ by considering the following dissection:

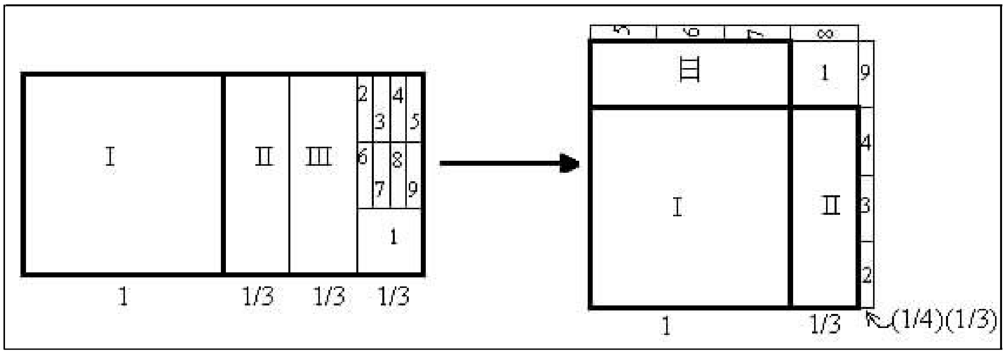

Figure 4

We now have that two square *pradesas* are equal to a large square minus a small square. The large square has side equal to 1 *pradesa* plus 1/3 of a *pradesa* plus 1/4 of 1/3 of a *pradesa*, or 1 *pradesa* and 5 *angulas* and the small square has side of 1 *angula*. To make this into a single square we may attempt to remove a thin strip from the left side and the bottom just thin enough that the strips will fill in the little square. Since these two thin strips will have length 1 *pradesa* and 5 *angulas* or 17 *angulas* we may cut each into 17 rectangular pieces each 1 *angula* long. If these are stacked up they will fill the little square if the thickness of the strips is 1/34 of an *angula* (or $\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ *pradesa*). Without a microscope we will now see the two square *pradesas* as being equal in area to the square with side $1+\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{3}-\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ *pradesa*. But with a microscope we see that the strips overlap in the lower left corner and thus that there is a tiny square of side $\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ still left out.

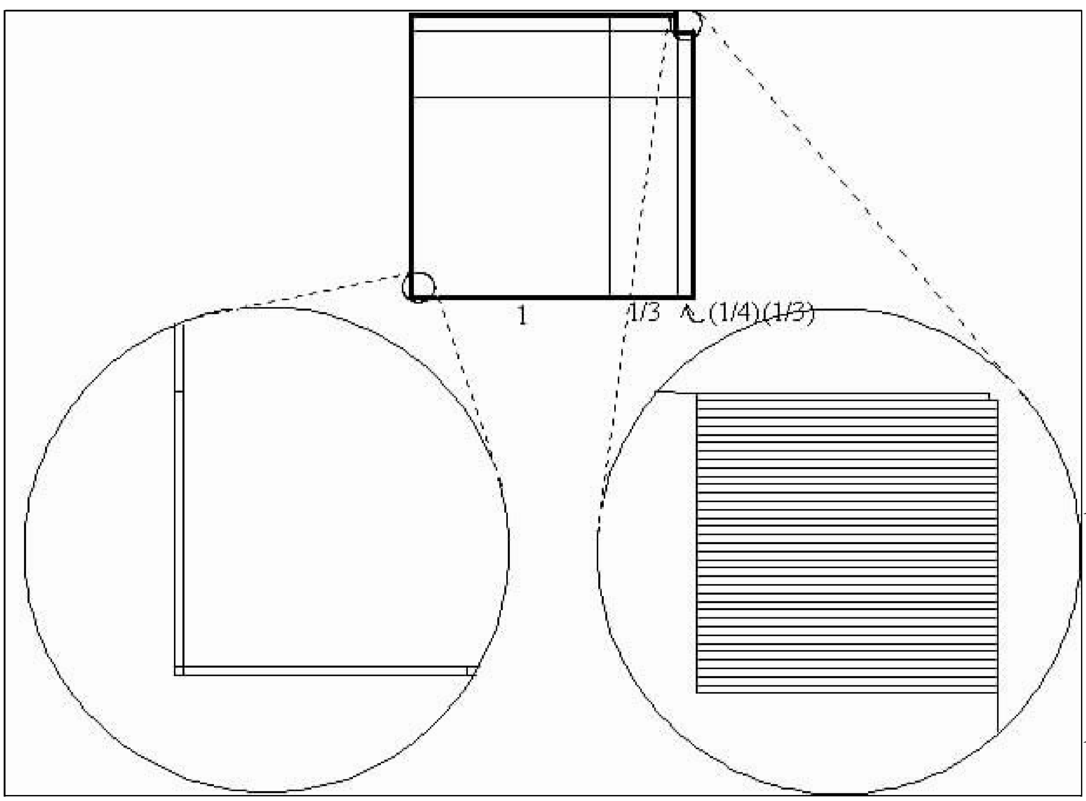

Figure 5

Thus $1+\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{3}-\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ is still a little in excess. We can now perform the same procedure again by removing a very very thin strip from the left and bottom edges and then cutting them into $\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ *pradesa* lengths in order to fill in the left out square. If *w* is twice the number of $\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ lengths in $1+\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{3}-\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3}$ *pradesa*, then the strips we remove must have width $\frac{1}{2w}\cdot(\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3})$ *pradesa*. We can calculate *w* easily because we already noted that there were 17 segments of length $\frac{1}{4}\cdot\frac{1}{3}$ in the length $1+\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{3}$ and each of these segments was divided into 34 pieces and then one of these pieces was removed. Thus $w = 2[34(17)-1] = 1154$ and

$$\sqrt{2} \approx 1 + \tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}$$

with error expressed by

$$2\cdot 1 = \left(1 + \tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}\right)^2 - \left(\tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}\right)^2.$$

I write "2·1" instead of "2" to remind us that for Baudhayana (and, in fact, for most mathematicians up until near the end of the 19th Century) that $\sqrt{2}$ denoted the side (a *length*) of a square with *area* 2.

If we again follow the same procedure of removing a very thin strip from the left and bottom edges and cutting them into $\tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}$ length pieces, then the reader can check that the number of such pieces must be

$$2[1154(1154/2)-1] = (1154)^2 - 2 = 1{,}331{,}714$$

and thus that the next approximation (*savi´e¸a*) is

$$\sqrt{2} \approx 1 + \tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3} - \tfrac{1}{1{,}331{,}714}\cdot\tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}.$$

The difference between 2·1 and the square of this *savi´e¸a* is

$$\left(\tfrac{1}{1{,}331{,}714}\cdot\tfrac{1}{1154}\cdot\tfrac{1}{34}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}\right)^2.$$

This method will work for any number N which you can first express as the area of the difference of two squares, $N\cdot 1 = A^2 - B^2$, where the side A is an integral multiple of the side B. For example,

$$5\cdot 1 = (2+\tfrac{1}{4})^2 - (\tfrac{1}{4})^2,\ 7\cdot 1 = (2+\tfrac{2}{3})^2 - (\tfrac{1}{3})^2,\ 10\cdot 1 = (3+\tfrac{1}{6})^2 - (\tfrac{1}{6})^2,$$

$$12\cdot 1 = (3+\tfrac{1}{2})^2 - (\tfrac{1}{2})^2,\ \text{and}\ (2+\tfrac{1}{2})\cdot 1 = (1+\tfrac{1}{2}+\tfrac{1}{6}\cdot\tfrac{1}{2})^2 - (\tfrac{1}{6}\cdot\tfrac{1}{2})^2.$$

I find that the easiest way for me to see that these expressions are valid is to represent them geometrically in a way that would also have been natural for Baudhayana. To illustrate:

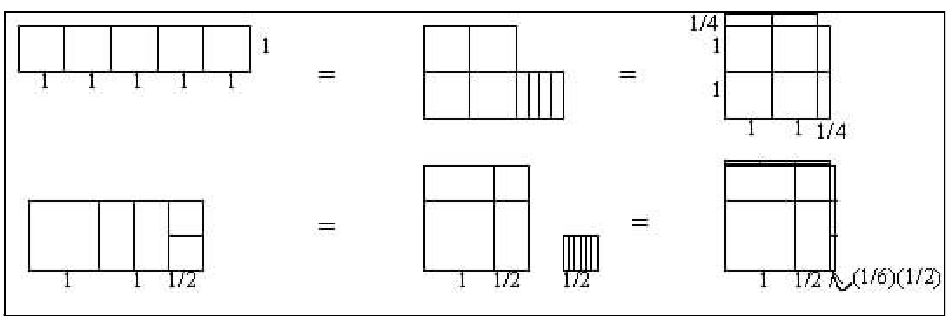

Figure 6

Figures 3 and 4 give other examples. The reader should try out this method to see how easy it is to find *savi´e¸as* for the square roots of other numbers, for example, 3, 11, 2¾.

Fractions in the Sulbasutram

You have probably noticed that all the fractions above are expressed as unit fractions, but this is not always the case in the Baudhayana's *Sulbasutram*. For example, in Sutra 69 he discusses how to find a length which is an approximation to the diagonal of a square whose side is the "third part of" 8 *prakramas* (which equals

240 *angulas*). He describes the construction:

... increase the measure [the 8 *prakramas*] by its fifth, divide the whole into five parts and make a mark at the end of two parts.

In more modern notation if we let D equal 8 *prakramas*, then this gives the approximation of the diagonal of a square with side (1/3)D as

$$\frac{2}{5}(D+\frac{1}{5}D).$$

This is equivalent to $\sqrt{2}$ being approximated by 1.44.

If you attempt to find the *savi´e¸as* for other square roots you will find it convenient to use non-unit fractions. For example, by starting with this picture:

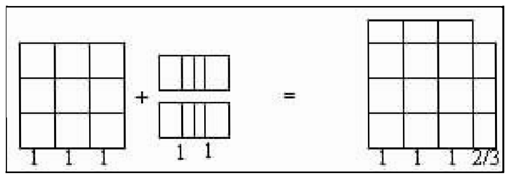

Figure 7

you can make slight modifications in the above method to find:

$$13\cdot 1 \approx \left(3+\frac{2}{3}-\frac{1}{11}\cdot\frac{2}{3}-\frac{1}{120}\cdot\frac{1}{11}\cdot\frac{2}{3}\right)^2-\left(\frac{1}{120}\cdot\frac{1}{11}\cdot\frac{2}{3}\right)^2.$$

Comparing with the Divide-and-Average (D&A) Method

Today the most efficient method usually taught to find square roots is called "divide-and-average". It is also sometimes called Newton's method. If you wish to find the square root of N then you start with an initial approximation a_0 and then take as the next approximation the average of a_0 and N/a_0. In general, if a_n is the n^{th} approximation of the square root of N, then $a_{n+1} = ½(a_n + (N/a_n))$. The interested reader can check that if you start with [1+(1/3)+(1/12)] = [17/12] = 1.416666666667 as your first approximation of $\sqrt{N}$, then the succeeding approximations are numerically the same as those given by Baudhayana's geometric method.

However, Baudhayana's method uses significantly less computations (in addition, of course, to the drawings either on paper or in one's mind). For example, look at the following table which compares the methods for the first four approximations. For Baudhayana's method at the n-th stage let k_n denote the number of thin pieces added into the missing square and let c_n denote the correction term that is added .

D&A - calculator	D&A - Fractions	Baudhayana's Method
a_1 = 1.416666667	17/12	1 + (1/3) + (1/4)(1/3)
$a_2 = ½(a_1 + (2/a_1))$ = 1.414215686	½[(17/12)+ 2(12/17)] = (577/408)	$k_2 = 2[(3\cdot4)+4+1] = 34$ $c_2 = -(1/34)(1/4)(1/3)$
$a_3 = ½(a_2 + (2/a_2))$ = 1.414213562	½[(577/408)+2(408/577)] = (665857/470832)	$k_3 = (34)^2-2 = 1154$ $c_3 = -(1/1154)(1/34)(1/4)(1/3)$

$a_4 = ½(a_3 + (2/a_3)) =$ 1.414213562	½[(665857/470832)+2(470832/665857)] = (886731088897/627013566048)	$k_4 = (1154)^2 - 2 = 1331714$ $c_4 = -(1/1331714)\, c_3$

Notice that the (10-digit) calculator reaches its maximum accuracy at the third stage. At this stage the Baudhayana method obtained more accuracy (it can be checked that it is accurate to 12-digits) and the only computation required was $(34)^2 - 2 = 1154$ which can easily be accomplished by hand. Baudhayana's approximations are numerically identical to those attained in the D&A method using fractions, but again with significantly less computations. Of course, Baudhayana's method has this efficiency only if you do not change Baudhayana's representation of the approximation into decimals or into standard fractions. At the fourth stage the Baudhayana method is accurate to less than $2[(1331714^2 - 2)(1331714)(1154)(34)(4)(3)]^{-1}$ or roughly 24-digit accuracy with the only calculation needed being $(1154)^2 - 2 = 1331714$.

Notice that in Baudhayana's fourth representation of the *savi´e¸a* for the square root of 2:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} - \frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3} - \frac{1}{1154}\cdot\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3} - \frac{1}{1{,}331{,}714}\cdot\frac{1}{1154}\cdot\frac{1}{34}\cdot\frac{1}{4}\cdot\frac{1}{3},$$

the unit is first divided into 3 parts and then each of these parts into 4 parts and then each of these parts into 1154 parts and each of these parts into 133174 parts. Notice the similarity of this to standard USA linear measure where a mile is divided into 8 furlongs and a furlong into 220 yards and a yard into 3 feet and a foot into 12 inches. Other traditional systems of units work similarly except for the metric systems where the division is always by 10. Also, some carpenters I know when they have a measurement of $2\frac{7}{16}$ inches are likely to work with it as $2 + \frac{1}{2} - \frac{1}{8}\frac{1}{2}$, or 2 inches plus a half inch minus an eighth of that half — this is a clearer image to hold onto and work with. From Baudhayana's approximation it is easier to have an image of the length of $\sqrt{2}$ than it is from the D&A's (886731088897/627013566048).

Conclusions

Baudhayana's method can not come even close to the D&A method in terms of ease of use with a computer and its applicability to finding the square root of any number. However, the *Sulbasutra* contains many powerful techniques, which, in specific situations have a power and efficiency that is missing in more general techniques. Numerical computations with the decimal system in either fixed point or floating point form has many well-known problems.[7] Perhaps we will be able to learn something from the (apparently) first applied geometry text in the world and devise computational procedures that combine geometry and numerical techniques.

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[1] This article grew out of researches which were started during my January, 1990, visit to the Sankaracharya Mutt in Konchipuram, Tamilnadu, India, where I was given access to the Mutt's library. I thank Sri Chandrasekharendra Sarasvati, the Sankaracharya, and all the people of the Mutt for their generous hospitality, inspiration and blessings.

[2] See for example, A. Seidenberg, The Ritual Origin of Geometry, *Archive for the History of the Exact Sciences*, 1(1961), pp. 488-527.

[3] *Baudhayana Sulbasutram, i. 61-2. Apastamba Sulbasutram, i. 6. Katyayana Sulbasutram, II. 13.*

[4] This last sentence is translated by some authors as "The increased length is called *savi´e¸a*". I follow the translation of "*savi´e¸a*" given by B. Datta on pp. 196-202 in *The Science of the Sulba*, University of Calcutta, 1932; see also G. Joseph (*The Crest of the Peacock,* I.B. Taurus, London, 1991) who translates the word as "a special quantity in excess".

[5] See Datta *Op.cit.* for a discussion of several of these, some of which are also discussed in G. Joseph, *Op. cit.*

[6] Baudhayana Sulbasutram, i. 3-7.

[7] See, for example, P.R. Turner's "Will the 'Real' Real Arithmetic Please Stand Up?" in *Notices of AMS*, Vol. 34, April 1991, pp. 298-304.