Proof of Kuṭṭaka's method

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Kuttaka: proof

We give below the justification of the kuṭṭaka's method.
Consider the equation: $8y = 29x + 4$

Dividend(bhājya) = 29, Divisor(hāra) = 8, kṣepaka(constant) = 4

Dividend	Divisor	Quotient	Remainder
29	8	3	5
8	5	1	3
5	3	1	2
3	2	1	1

Valli:
Place the quotients, constant and zero in a column.

3	3				44(=y)
1	1			12	12(=x)
1	1		8	8	
1	1	4	4		
4	4	4			
0	0				

Kuttaka: proof

The equation is:

$8Y = 29X + 4$

$8Y_1 = 5X + 4$ where $Y_1 = Y - \mathbf{3}X$

$3Y_1 = 5X_1 + 4$ where $X_1 = X - \mathbf{1}Y_1$

$3Y_2 = 2X_1 + 4$ where $Y_2 = Y_1 - \mathbf{1}X_1$

$Y_2 = 2X_2 + 4$ where $X_2 = X_1 - \mathbf{1}Y_2$

The equation is:
$8Y = 29X + 4$

$8Y_1 = 5X + 4$	$Y_1 = Y - \mathbf{3}X$	$Y = Y_1 + 3 * X$
$5X_1 = 3Y_1 - 4$	$X_1 = X - \mathbf{1}Y_1$	$X = X_1 + 1 * Y_1$
$3Y_2 = 2X_1 + 4$	$Y_2 = Y_1 - \mathbf{1}X_1$	$Y_1 = Y_2 + 1 * X_1$
$2X_2 = Y_2 - 4$	$X_2 = X_1 - \mathbf{1}Y_2$	$X_1 = X_2 + 1 * Y_2$

Let $X_2 = 0$. So $Y_2 = 4$

$Y = Y_1 + 3 * X$ $X = X_1 + 1 * Y_1$ $Y_1 = Y_2 + 1 * X_1$ $X_1 = X_2 + 1 * Y_2$	$Y = 8 + 3 * 12 = 44$ $X = 4 + 1 * 8 = 12$ $Y_1 = 4 + 1 * 4 = 8$ $X_1 = 0 + 1 * 4 = 4$
$Y_2 = 4$	$Y_2 = 4$
$X_2 = 0$	$X_2 = 0$

General Proof

Let the equation be $ay = bx + c$ with b $>$ a $>$ 0

Let $b = aq_1 + r_1$
Hence, $ay = (aq_1 + r_1)x + c$
i.e., $a(y - q_1x) = r_1x + c$

Let $y - q_1x = X_1$
Hence $aX_1 - c = r_1x$

Let $a = r_1q_2 + r_2$
Hence $(r_1q_2 + r_2)X_1 - c = r_1x$
or, $r_2X_1 = r_1(x - q_2X_1) + c$

Let $x - q_2X_1 = X_2$
Hence $r_2X_1 = r_1X_2 + c$

General Proof ... Contd

Continuing in this way,
$r_{k-1}X_k = r_kX_{k-1} + (-1)^{k-1}c$, where
$X_k = X_{k-2} - q_kX_{k-1}$

Finally, we stop when $r_n = 1$.

Now let $X_n = 0$ in
$r_{n-1}X_n = r_nX_{n-1} + (-1)^(n-1)c$

Then we have,
$X_{n-1} = (-1)^n c$

Substituting the values of X_i we in fact end up creeping up as shown below.

y	q_1	
x	q_2	
x_1	q_3	
.	.	.
.	.	.
.	.	.
x_{n-3}	q_{n-1}	$x_{n-3} = q_{n-1} * x_{n-2} + c$
x_{n-2}	q_n	$x_{n-2} = q_n * c + 0$
x_{n-1}	c	
x_n	0	

General Proof ... Contd

The solution we get is for $ax + c = by$, if there are even number of quotients (rows).

But if there are odd number of rows, this is the solution for $ax - c = by$.

Now,
$a(b - x) + c = ab - ax + c = ab - (by + c) + c = b(a - y)$

Hence, $(b - x, a - y)$ is the solution of $ax + c = by$.

This justifies Bhaskaracharya's corollory for handling odd number of quotients.