# On the Construction of Śivasūtra-Alphabets

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**Abstract.** In the present paper, a formalization of the technique used by Pāṇini in his Śivasūtras for the denotation of sound classes is given. Furthermore, a general notion of Śivasūtra-alphabets and of Śivasūtra-sortability is developed. The presented main theorem poses three sufficient conditions for the Śivasūtra-sortability of sets of classes. Finally, the problem of ordering sets of classes which are not Śivasūtra-sortable is tackled and an outlook on modern problems which could be approached by Pāṇini's technique is given.

Keywords: Śivasūtras, Pāṇini, orders, lattices

## 1 Introduction

### 1.1 Pāṇini's Śivasūtra-technique

Among linguists Pāṇini's grammar of Sanskrit is acknowledged to be the culmination point of ancient Indian grammar:

> Indian linguistics originated among reciters who wanted to preserve their Vedic heritage and apply it in ritual. Unconcerned with meaning, they concentrated on form and incorporated a good measure of linguistic analysis that culminated in the Sanskrit grammar of Pāṇini. (**?**)

Although more than 2 000 years old, Pāṇini's grammar is rather accurately preserved due to the fact that it was soon considered to be the standard grammar of Sanskrit. Thereby the originally descriptive grammar of a living language achieved the status of a normative, prescriptive grammar (cf. **?**). Situated in the oral culture of ancient India, Pāṇini's grammar was designed for repetitive recitation. Thus the grammar is necessarily presented in a purely linear form, and its compactness was particularly desirable.

Its main part consists of about 4 000 rules, many of them phonological rules which describe the complex system of Sanskrit Sandhi (cf. **?**). Phonological rules are typically of the form "sounds of class $A$ are replaced by sounds of class $B$ if they are preceded by sounds of class $C$ and followed by sounds of class $D$", which in modern phonology is usually denoted as

$$A \to B/_{C\_\_D}\ . \qquad (1)$$

A;I+.o+.N,a ‘x+.‘w+.k̦ O;A:ea;z̦ Oe;A:Ea;.c,a h;ya;va:=+,f l+.N,a Va;ma;z+.Na;na;m,a Ja;Ba;V,a (I)
;Ga;Q+Da;S,a .ja;ba;ga;q+.d;Z,a Ka;P+.C+.F+.Ta;.ca;f;ta;v,a k+:pa;y,a Za;Sa;sa:=, h;l̦

a·i·uṇ ṛ·ḷk e·oṅ ai·auc hayavaraṭ laṇ ñamaṅaṇanam jhabhañ (II)
ghaḍhadhaṣ jabagaḍadaś khaphachaṭhathacaṭatav kapay śaṣasar hal

**Fig. 1.** Pāṇini's Śivasūtras in linear form ( I: Devanāgarī script; II: Latin transcription)

Since Pāṇini's grammar has been designed for oral tradition, it makes no use of visual symbols (like arrows, slashes ...) to indicate the role of the sound classes in a rule. Instead, Pāṇini takes natural case suffixes which he uses meta-linguistically in a formalized way in order to mark the role a class plays in a rule. In Pāṇinian style rule (1) becomes

$$A + \text{genitive}, B + \text{nominative}, C + \text{ablative}, D + \text{locative}\ . \qquad (2)$$

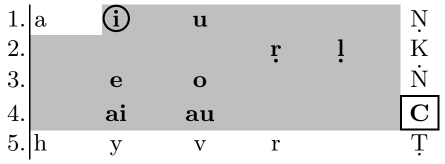
Since constantly repeating the single sounds of each class involved in a rule is not economical, an appropriate phonological description must involve a method to denote the sound classes. The method should be such that it is easier to address a natural phonological class than an arbitrary set of sounds (cf. **??**). A wide-spread technique in modern phonology is to build up a structured system of phonetic features (e.g., [±consonantal] or [±voiced]) in order to define the phonologically relevant sound classes. The aim is to identify phonetic features, i.e., features that are motivated by properties of the isolated sounds, by which the sounds can be classified into phonological classes, i.e., into classes of sounds with analogous behavior in the same speech contexts. Unfortunately, this leads to the problem of choosing and naming features and often involves the danger of defining ad-hoc features.

Pāṇini's technique for the denotation of sound classes allows him to do completely without features. His grammar of Sanskrit begins with 14 sūtras, the so-called *Śivasūtras*, which are quoted in Fig. 1 in their original linear form and in Fig. 2 in the tabular form given in **?**. Each single sūtra consists of a sequence of sounds which ends in a consonant, the so-called *anubandha*. This last consonant of each sūtra is used meta-linguistically as a marker to indicate the end of a sūtra. According to **?** the system behind the choice of the consonants used as anubandhas is unknown. Together the Śivasūtras define a linear list of Sanskrit sounds which is interrupted by marker elements (anubandhas). In his grammar Pāṇini uses *pratyāhāras*, i.e., pairs consisting of a sound and an anubandha in order to designate the sound classes on which a rule operates. Such a pair denotes the sounds in the interval between the sound and the anubandha; e.g., the pratyāhāra iC denotes the class {i, u, ṛ, ḷ} as depicted in Fig. 3.[1] Pratyāhāras are often used in rules of type (2) where they replace the open place-holders $A$, $B$, $C$ and $D$.

[1] To simplify matters we ignore here that a pratyāhāra actually denotes the ordered list of sounds in the interval and not just the unordered class of sounds.

|  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| 1. | a | i | u |  |  | Ṇ |
| 2. |  |  |  | ṛ | ḷ | K |
| 3. |  | e | o |  |  | Ṅ |
| 4. |  | ai | au |  |  | C |
| 5. | h | y | v | r |  | Ṭ |
| 6. |  |  |  |  | l | Ṇ |
| 7. | ñ | m | ṅ | ṇ | n | M |
| 8. | jh | bh |  |  |  | Ñ |
| 9. |  |  | gh | ḍh | dh | Ṣ |
| 10. | j | b | g | ḍ | d | Ś |
| 11. | kh | ph | ch | ṭh | th |  |
|  |  |  | c | ṭ | t | V |
| 12. | k | p |  |  |  | Y |
| 13. |  |  | ś | ṣ | s | R |
| 14. | h |  |  |  |  | L |

**Fig. 2.** Pāṇini's Śivasūtras in tabular form (the default, syllable-building vowel *a* is left out)



|  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| 1. | a | ⓘ | **u** |  |  | Ṇ |
| 2. |  |  |  | **ṛ** | **ḷ** | K |
| 3. |  | **e** | **o** |  |  | Ṅ |
| 4. |  | **ai** | **au** |  |  | **C** |
| 5. | h | y | v | r |  | Ṭ |

**Fig. 3.** example of a pratyāhāra: iC = {i,u,ṛ,ḷ,e,o,ai,au}

There is a longstanding debate on how Pāṇini developed the Śivasūtras and whether he arranged the sounds in the best way possible (cf. **????**). Note that exactly one sound, namely *h*, occurs twice in the list of the Śivasūtras (in the 5th and in the 14th sūtra). Nowadays it is generally assumed that the order of the sounds in the Śivasūtras is primarily determined by the structural behavior of the sounds in the rules of Pāṇini's grammar and that the arrangement of the sounds is chosen such that economy or rather brevity is maximized (cf. **????**). **?** argues "that the structure of the Śivasūtras is entirely explicable on systematic grounds [. . . and] that no other principles are needed than those used in the construction of the rest of Pāṇini's grammar, namely the principle of economy and the logic of the special case and the general case."

In **?** it has been formally proven that there is no shorter solution than the Śivasūtras to the problem of ordering the sounds of Sanskrit in a by markers interrupted, linear list with as few repeated sounds as possible such that each phonological class which is denoted by a sound-marker pair (i.e., a pratyāhāra) in Pāṇini's grammar can be represented by such a pair with respect to the list. Hence, Pāṇini was forced to duplicate one sound, namely *h*, in the Śivasūtras and he used a minimal number of markers. Actually, it can be shown that there are nearly 12 000 000 alternative sound lists interrupted by markers which fulfill the above mentioned conditions and which are of the same length as the Śivasūtras (**?**). The question whether the actual list chosen by Pāṇini in the Śivasūtras results, as **?** argues, from the 'principle of economy' and the 'logic of the special case and the general case' and not from the 'principle of historic continuity' (**?**) or the 'principle of homorganic continuity' (**?**) cannot be answered by the mathematical reasoning in **?**.

The present paper focuses not so much on the concrete list of the Śivasūtras as former ones (cf. **??**), but concentrates more on the general technique of ordering entities in a list which is interrupted by marker elements such that each class of entities out of a given set of classes forms an interval and thus can be unambiguously addressed by a pair consisting of an entity and a marker. In particular it is examined under which conditions it is possible to construct such a list without being forced to include an entity twice.

### 1.2 General problem of S-sortability

As a start we will simplify Pāṇini's Śivasūtra-technique by abandoning the claim that the target list is interrupted by markers and that each class which is denotable with respect to the list forms an interval which ends immediately before a marker. Thus the simplified problem states as follows:

*Problem 1.* Given a set of classes, order the elements of the classes in a linear order such that each single class forms a continuous interval with respect to that order.

The target orders will be called S-orders:

**Definition 1.** *Given a finite base set $\mathcal{A}$ and a set of subsets $\Phi$ with $\bigcup \Phi = \mathcal{A}$, a linear order $<$ on $\mathcal{A}$ is called a* Śivasūtra-order *(or short* S-order*) of $(\mathcal{A}, \Phi)$ if and only if the elements of each set $\phi \in \Phi$ form an interval in $(\mathcal{A}, <)$, i.e., $\forall \phi \in \Phi$ : if $\phi_{min}$ is the minimum of $\phi$ w.r.t. $(\mathcal{A}, <)$ and $\phi_{max}$ is the maximum of $\phi$, then there is no $a \in \mathcal{A} \setminus \phi$ s.th. $\phi_{min} < a < \phi_{max}$.*

*Furthermore, $(\mathcal{A}, \Phi)$ is said to be* S-sortable *if and only if there exists an S-order $(\mathcal{A}, <)$ of $(\mathcal{A}, \Phi)$.*

*Example 1.* Given the base set $\mathcal{A} = \{a, b, c, d, e, f, g, h, i\}$ and the set of classes $\Phi = \{\{d, e\}, \{a, b\}, \{b, c, d, f, g, h, i\}, \{f, i\}, \{c, d, e, f, g, h, i\}, \{g, h\}\}$, $(\mathcal{A}, \Phi)$ is S-sortable and $a \prec b \prec c \prec g \prec h \prec f \prec i \prec d \prec e$ is an S-order of $(\mathcal{A}, \Phi)$.[2]

It is important to note that not all sets of classes are S-sortable. For instance, since the duplication of at least one sound element in the Śivasūtras is unavoidable, the set of classes defined by the sound classes in Pāṇini's grammar which are denoted by pratyāhāras is not S-sortable. Orders, like the one underlying the Śivasūtras, which contain at least one element twice will be called S-orders *with duplications*. A smaller example of a non S-sortable set of classes is given here:

[2] As usual, $\prec$ stands for the binary predecessor relation, i.e., $a \prec b$ if and only if $a < b$ and there is no $c$ such that $a < c < b$.

*Example 2.* Given the base set $\mathcal{A} = \{a, b, c, d, e, f\}$ and the set of classes $\Phi = \{\{d, e\}, \{a, b\}, \{b, c, d\}, \{b, c, d, f\}\}$, $(\mathcal{A}, \Phi)$ is not S-sortable (without duplications).

One of the major aims of the present paper is to examine the conditions which S-sortable sets of classes fulfill and to show how these conditions can be constructively applied to different tasks: (1) the building of concrete S-orders, (2) the identification of best candidates for duplication in the case of non S-sortable sets of classes, (3) the insertion of a minimal amount of marker elements such that each class forms an interval that ends immediately before a marker. S-orders which are interrupted by marker elements are called S-alphabets and defined as follows:

**Definition 2.** *Given a finite base set $\mathcal{A}$ and a set of subsets $\Phi$ with $\bigcup \Phi = \mathcal{A}$, a* Śivasūtra-alphabet *(short* S-alphabet*) of $(\mathcal{A}, \Phi)$ is a triple $(\mathcal{A}, \Sigma, <)$ with*

- *$\Sigma$ is a finite set of markers with $\mathcal{A} \cap \Sigma = \emptyset$,*
- *$<$ is a linear order on $\mathcal{A} \cup \Sigma$*

*if and only if for each $\phi \in \Phi$ there exists $a \in \phi$ and $M \in \Sigma$ such that $\phi = \{b \in \mathcal{A} \mid a \leq b < M\}$ ($aM$ is called the pratyāhāra or* S-encoding *of $\phi$).*

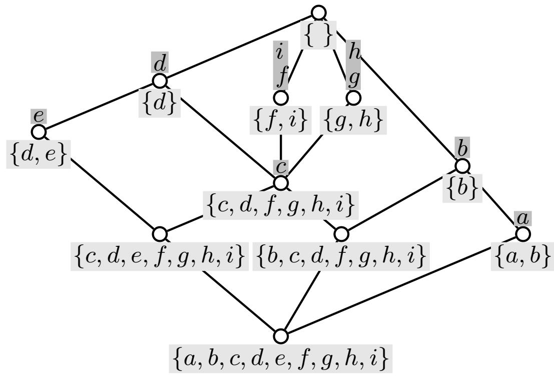
*Furthermore, $(\mathcal{A}, \Phi)$ is said to be* S-encodable *if and only if there exists an S-alphabet $(\mathcal{A}, \Sigma, <)$ of $(\mathcal{A}, \Phi)$.*

It follows from definition 2 that whenever $(\mathcal{A}, \Sigma, <)$ is an S-alphabet of $(\mathcal{A}, \Phi)$ then $(\mathcal{A}, < |_{\mathcal{A}})$ is an S-order of $(\mathcal{A}, \Phi)$. Furthermore, since every S-order can be trivially enhanced into an S-order by inserting a marker behind each element, it is true that each S-sortable set of classes is S-encodable and vice versa.

## 2 Main theorem on S-sortability

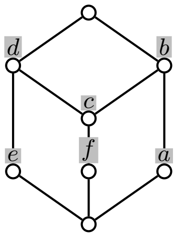
The main theorem on S-sortability depends on two constructs taken from Formal Concept Analysis (FCA), which is a mathematical theory for the analysis of data (cf. **?**). For our purposes, we do not need to evolve the whole apparatus of FCA, it is sufficient to define what we understand by the formal context and the concept lattice of a set of classes $(\mathcal{A}, \Phi)$: Given a base set $\mathcal{A}$ and a set of subsets $\Phi$, the *formal context* of $(\mathcal{A}, \Phi)$ (or the $(\mathcal{A}, \Phi)$-context) is the triple $(\Phi, \mathcal{A}, \ni)$ and the concept lattice of $(\mathcal{A}, \Phi)$ (or the $(\mathcal{A}, \Phi)$-lattice) is the ordered set $(\mathcal{A} \cup \{\psi \mid \psi = \bigcap \Psi \text{ with } \Psi \subseteq \Phi\}, \supseteq)$.

Given the base set $\mathcal{A}$ and the set of classes $\Phi$ in example 1, the formal context of $(\mathcal{A}, \Phi)$ and the Hasse-diagram of the concept lattice of $(\mathcal{A}, \Phi)$ are depicted in Fig. 4. The formal context is given in form of a cross table as usual. Its concept lattice is constructed as follows: All elements of $\Phi$ and all possible intersections of elements of $\Phi$ are ordered by the set-inclusion relation such that subsets are placed above their supersets. The *Hasse-diagram* of an ordered set is the directed graph whose vertices are the elements of the set and whose edges correspond to the upper neighbor relation determined by the order. An ordered set is said to



| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|---|
| $\{d, e\}$ | | | | × | × | | | | |
| $\{b, c, d, f, g, h, i\}$ | | × | × | × | × | × | × | × | × |
| $\{a, b\}$ | × | × | | | | | | | |
| $\{f, i\}$ | | | | | | × | | | × |
| $\{c, d, e, f, g, h, i\}$ | | | × | × | × | × | × | × | × |
| $\{g, h\}$ | | | | | | | × | × | |

**Fig. 4.** concept lattice (left) and formal context (right) of $(\mathcal{A}, \varPhi)$ in example 1



| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $\{d, e\}$ | | | | × | × | |
| $\{a, b\}$ | × | × | | | | |
| $\{b, c, d\}$ | | × | × | × | | |
| $\{b, c, d, f\}$ | | × | × | × | | × |

**Fig. 5.** concept lattice (left) and formal context (right) of $(\mathcal{A}, \varPhi)$ in example 2

be Hasse-planar if its Hasse-diagram can be drawn without intersecting edges. Hence, the concept lattice in Fig. 4 is Hasse-planar.

The node labeling in Fig. 4 is twofold: The labels below the nodes indicate the corresponding sets. For the labels above the nodes a more economic labeling is chosen which assigns each element of the base set $\mathcal{A}$ to the node corresponding to the smallest set which contains the element. The labels below the nodes are superfluous as they can be reconstructed from the others by collecting all labels attached to nodes which can be reached by moving along paths upwards in the graph. From now on, solely the upper labels will be used in figures of concept lattices, as seen in Fig. 5, which shows the concept lattice for example 2.

The main theorem on S-sortability states three equivalent, sufficient conditions which a set of classes must fulfill in order to be S-sortable. The individual conditions will be explained in detail in the succeeding subsections.

**Theorem 1.** *A set of classes $(\mathcal{A}, \varPhi)$ is S-sortable if and only if one of the following equivalent statements is true:*

*Condition 1:* *Let $\tilde{\varPhi} = \varPhi \cup \{\{a\} \mid a \in \mathcal{A}\}$. The concept lattice of the enlarged set of classes $(\mathcal{A}, \tilde{\varPhi})$ is Hasse-planar.*

*Condition 2:* *The concept lattice of $(\mathcal{A}, \varPhi)$ is Hasse-planar and for any $a \in \mathcal{A}$ there is a node labeled $a$ in the S-graph of the concept lattice.*

*Condition 3:* *The Ferrers-graph of the enlarged $(\mathcal{A}, \tilde{\varPhi})$-context is bipartite.*

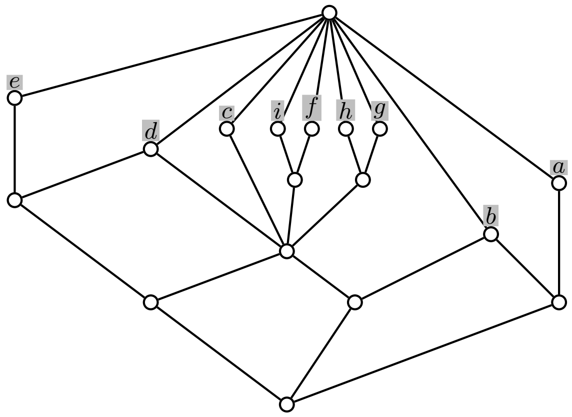


**Fig. 6.** enlarged concept lattice for example 1

Although all three conditions depend on properties of graphs, they are of different nature. The first one demands that the Hasse-diagram of a concept lattice can be drawn without intersecting edges; the second one relies on the positions of certain labels in such a Hasse-diagram; and the third one depends on the bipartity of so-called Ferrers-graphs. Instead of giving the proof of the theorem in isolation, the following subsections treat the three conditions for S-sortability one by one. For each condition, illustrational examples are given, a proof of its sufficiency is sketched, and it is demonstrated how the condition can be applied in the construction of S-alphabets with as few duplicated elements as possible and a minimal number of markers.

### 2.1 First Condition for S-sortability: Main planarity criterion

Condition 1 relates S-sortability with Hasse-planarity of enlarged concept lattices. Here, a set of classes gets enlarged by adding each element of the base set as a singleton set to the set of classes, e.g., in the case of example 1 the classes $\{a\}$, $\{b\}$, $\{c\}$, ..., $\{i\}$ have to be added. The condition states that a set of classes is S-sortable if and only if the concept lattice of the so enlarged set of classes is Hasse-planar, i.e., if it is possible to draw its Hasse-diagram without intersecting edges. Figure 6 shows a plane drawing of the enlarged concept lattice for the set of classes taken from example 1.

Figure 7 shows the Hasse-diagram of the enlarged concept lattice that belongs to the set of classes in example 2 which is not S-sortable. In the case of this small lattice it can be easily verified that it is impossible to draw its Hasse-diagram without intersecting edges.

Condition 1 is proven in detail in **?**. The fact that the existence of a plane drawing of the Hasse-diagram of an enlarged concept lattice implies the existence of an S-order follows immediately from the definition of our concept lattices: Since concept lattices order sets by set inclusion it is ensured that in the case

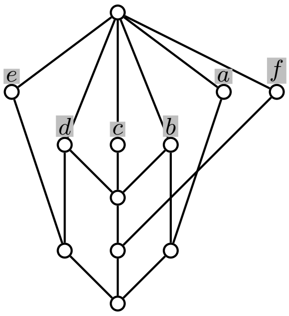


**Fig. 7.** enlarged concept lattice for example 2

of an enlarged set of classes the labels belonging to the elements of one class form an interval in the sequence defined by the left-to-right order of the labels in a plane drawing of the Hasse-diagram of the concept lattice. It can be easily seen that this guarantees that the left-to-right order of the labels in a plane drawing of the Hasse-diagram of a concept lattice of an enlarged set of classes $(\mathcal{A}, \tilde{\Phi})$ defines an S-order of $(\mathcal{A}, \Phi)$. For instance, the S-order defined by the plane Hasse-diagram in Fig. 6 is $e \prec d \prec c \prec i \prec f \prec h \prec g \prec b \prec a$.

The second statement, i.e. that the existence of an S-order implies the existence of a plane drawing of the enlarged Hasse-diagram, was first proven in **?**. The proof is based on the controlled construction of a drawing of the enlarged concept lattice for each S-order which ensures that the drawing is plane. The resulting drawing is such that the left-to-right order of the labels equals the original S-order.

Since several plane drawings leading to different S-orders usually exist for a concept lattice of a set of classes, our construction method does not deterministically result in one S-order. In fact, the proof of the theorem above implies that for every S-order there exists a plane drawing of the concept lattice from which it can be read off. For instance, for the Hasse-diagram in Fig. 6 one finds 48 plane drawings leading to 48 distinct S-orders of the set of classes taken from example 1.

See **?** for a discussion on why and how S-orders can be fruitfully applied to the problem of ordering books in a library or products in a warehouse. In short, the applicability of S-orders to these problems is based on the fact that in S-orders elements belonging to one class are placed in close distance to each other.

As demonstrated, condition 1 reduces the problem of S-sortability nicely to the Hasse-planarity of certain concept lattices. However, in practice condition 1 is problematic for two reasons in particular: First, the condition is non-constructive for the problem of inducing S-alphabets with minimal marker sets. If one finds a plane drawing of the Hasse-diagram of an enlarged set of classes, it is always possible to read off an S-order, but usually not an S-alphabet of the original set of classes without superfluous markers. Each S-order can be trivially completed into an S-alphabet by inserting a marker element behind each element in the S-order, but such an S-alphabet will usually contain unnecessarily many markers. The problem is that by enlarging a set of classes the information about which elements do not need to be separated by a marker in an S-alphabet gets lost. Condition 2, which operates on concept lattices that are not enlarged, offers a way out of the dilemma.

Second and even worse, condition 1 does not offer an easily verifiable criterion for the S-sortability of a set of classes. The problem of determining whether a plane drawing of a general graph exists is hard. In section 2.3, which treats condition 3, a sufficient criterion for the Hasse-planarity of concept lattices will be presented which can be algorithmically checked.

## 2.2 Second condition for S-sortability: Minimizing the number of marker elements

Condition 2 consists of two parts. It states that a set of classes $(\mathcal{A}, \Phi)$ is S-sortable if and only if the following two conditions are fulfilled:

1. The concept lattice of $(\mathcal{A}, \Phi)$ is Hasse-planar.
2. For any $a \in \mathcal{A}$ there is a node labeled $a$ in the S-graph of the concept lattice of $(\mathcal{A}, \Phi)$.

The second part depends on the notion of S-graphs of concept lattices. S-graphs only exist for Hasse-planar concept lattices since their definition is based on plane drawings of Hasse-diagrams: Given a plane drawing of the Hasse-diagram of an $(\mathcal{A}, \Phi)$-lattice, remove the top node and all adjoined edges if it corresponds to the empty set (if the top node does not correspond to the empty set, do not change the drawing). The resulting drawing defines a plane graph, and the boundary graph of the infinite face of this graph is the S-graph of the $(\mathcal{A}, \Phi)$-lattice. In **?** it has been proven that for each S-sortable set of classes there exists exactly one S-graph up to isomorphism. Examples of S-graphs are given in Fig. 8.

The proof of condition 2 is based on the following considerations: According to condition 1 a set of classes $(\mathcal{A}, \Phi)$ is S-sortable if and only if the enlarged $(\tilde{\mathcal{A}}, \Phi)$-lattice is Hasse-planar. Since an S-order of $(\mathcal{A}, \Phi)$ is necessarily an S-order of $(\tilde{\mathcal{A}}, \Phi)$ too, it follows that the S-sortability of a set of classes implies the Hasse-planarity of its concept lattice. Hence, for any S-sortable set of classes a plane drawing of the Hasse-diagram of its concept lattice exists which implies the existence of the S-graph of its concept lattice. However, the Hasse-planarity is only a necessary, but not a sufficient precondition for S-sortability as Fig. 5 demonstrates, which shows a plane drawing of the Hasse-diagram of a concept lattice of a set of classes that is not S-sortable.

A close investigation of what happens while reducing a plane drawing of an enlarged concept lattice to a plane drawing of the non-enlarged concept lattice will conclude the proof of condition 2: A plane drawing of the non-enlarged concept lattice can be gained from a plane drawing of the enlarged one by contracting all edges leading from lower nodes to nodes which correspond to singleton sets which were added while enlarging the set of classes. For each such node

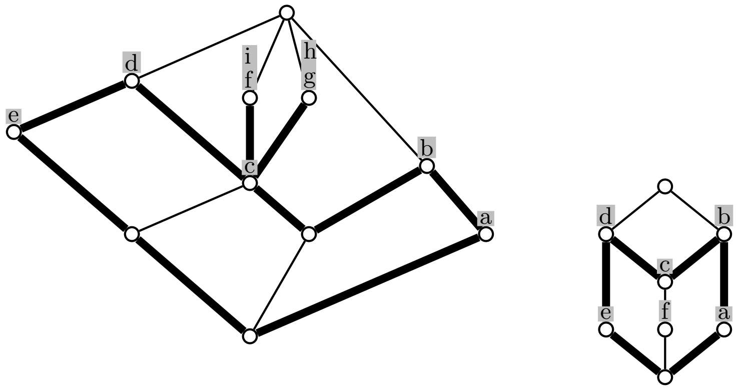


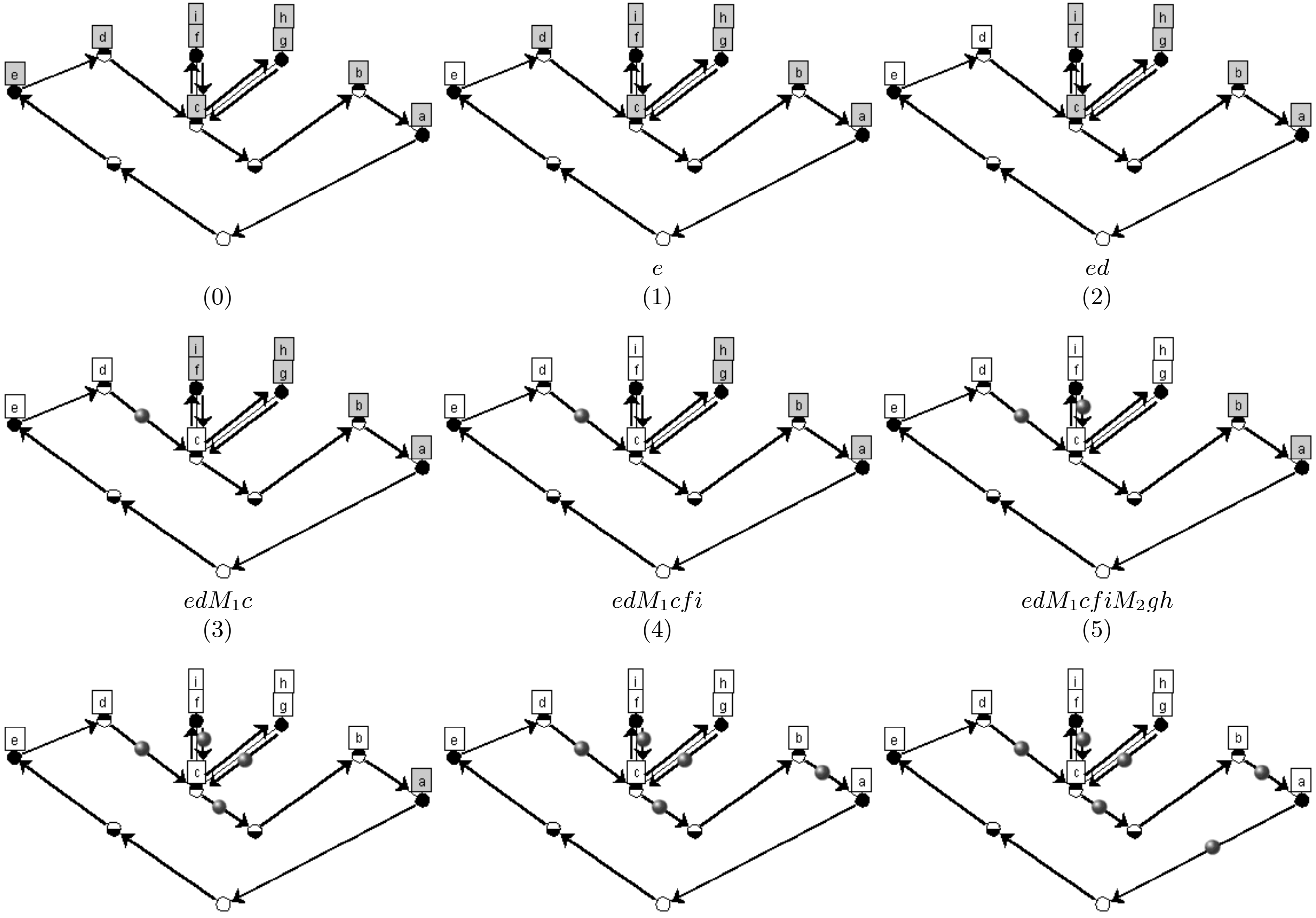
**Fig. 8.** S-graphs of $(\mathcal{A}, \Phi)$-lattices (left: $(\mathcal{A}, \Phi)$ taken from example 1, right: $(\mathcal{A}, \Phi)$ taken from example 2)

there will be exactly one edge which must be contracted. Hence, the contraction will not destroy the planarity of the graph. Furthermore, a node labeled $a$ (with $a \in \mathcal{A}$) which trivially belongs to the S-graph of the enlarged concept lattice will also belong to the S-graph of the non-enlarged concept lattice. This proves that condition 2 is equivalent to condition 1 and thus that it is a sufficient condition for the S-sortability of a set of classes.

In contrast to condition 1, condition 2 operates immediately on the concept lattice of the original set of classes. Hence, on the concept lattice which involves only those sets for which a pratyāhāra must exist in a corresponding S-alphabet. Therefore, it is possible to develop a procedure for the construction of S-alphabets with minimal marker sets on the basis of the S-graphs treated in condition 2. For a detailed illustration of how the procedure works see Fig. 9 which stepwise illustrates the procedure for the S-sortable set of classes given in example 1.

*Procedure for the construction of S-alphabets with minimal marker sets:*

1. Start with the empty sequence and choose a walk through the S-graph that:
   - starts and ends at the lowest node,
   - reaches every node of the S-graph,
   - passes each edge not more often than necessary,
   - is oriented such that while moving downwards as few labeled nodes with exactly one upper neighbor as possible are passed.
2. While walking through the S-graph modify the sequence as follows:
   - While moving upwards along an edge do not modify the sequence.
   - While moving downwards along an edge add a new marker to the sequence unless its last element is already a marker.
   - If a labeled node is reached, add the labels in arbitrary order to the sequence, except for those labels which have already been added in an earlier step.



(0)

$e$

(1)

$ed$

(2)

$edM_1c$

(3)

$edM_1cfi$

(4)

$edM_1cfiM_2gh$

(5)

$edM_1cfiM_2ghM_3b$

(6)

$edM_1cfiM_2ghM_3bM_4a$

(7)

$edM_1cfiM_2ghM_3bM_4aM_5$

(8)

**Fig. 9.** sequence of figures to illustrate the procedure for the construction of S-alphabets with minimal marker sets

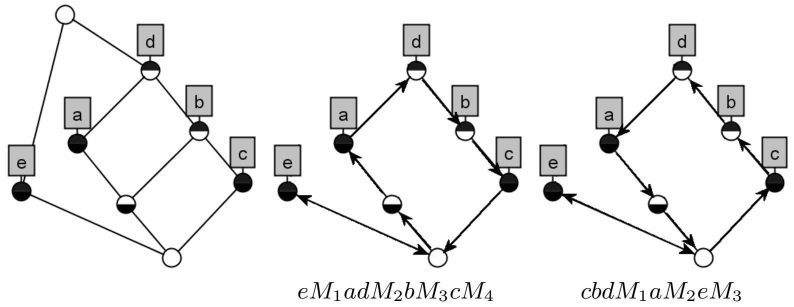


**Fig. 10.** example with two distinct walks through the S-graph of which only the right one leads to an S-alphabet with minimal marker set (here, $\Phi = \{\{a,d\},\{a,b,d\},\{b,c,d\},\{e\}\}$)

The small example given in Fig. 10 illustrates the importance of choosing a walk through the S-graph that avoids passing labeled nodes while moving downwards. In **?** a similar procedure is applied in order to demonstrate that the number of markers in Pāṇini's Śivasūtras cannot be reduced. Note that the procedure is not deterministic, as it usually does not return one single S-alphabet. In **?** it has been proven that every S-alphabet with minimal marker set can be derived by this procedure. In the case of Pāṇini's problem the procedure leads to nearly 12 000 000 equally short S-alphabets in which the sound $h$ occurs twice.

### 2.3 Third condition for S-sortability: Algorithmically verifiable criterion

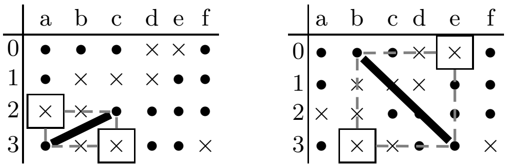
Since it is hard to decide whether a concept lattice is Hasse-planar by examining the concept lattice itself, it is favorable to use a planarity criterion which does not depend on properties of concept lattices (like condition 1 and 2), but on properties of their corresponding formal contexts which can be checked more easily. Condition 3 follows immediately from condition 1 and the following proposition which is proven in **?**:

**Proposition 1.** *The concept lattice of a formal context is Hasse-planar if and only if its Ferrers-graph is bipartite.*

A graph is said to be *bipartite* if it is possible to assign its vertices to two disjoint classes such that each edge connects vertices which belong to distinct classes. **?** defines the Ferrers-graph of a formal context as follows:

**Definition 3.** *The* Ferrers-Graph *of a formal context* $(G, M, I)$ *is* $\Gamma(I)$ *with*

$$\begin{array}{ll} \textit{set of vertices:} & V(\Gamma(I)) = \bar{I} \quad \textit{with} \quad \bar{I} = G \times M \setminus I \textit{ and} \\ \textit{set of edges:} & E(\Gamma(I)) = \{\{(a_1, b_2), (a_2, b_1)\} \mid (a_1, b_1), (a_2, b_2) \in I\} \,. \end{array}$$



|   | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| 0 | • | • | • | × | × | • |
| 1 | • | × | × | × | • | • |
| 2 | × | × | • | • | • | • |
| 3 | • | × | × | • | • | × |

|   | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| 0 | • | • | • | × | × | • |
| 1 | • | × | × | × | • | • |
| 2 | × | × | • | • | • | • |
| 3 | • | × | × | • | • | × |

**Fig. 11.** illustration for the definition of edges in Ferrers-graphs

The definition is easier to understand if one describes the formal context by a cross table as before (cf. Fig. 4 and Fig. 5). Then the empty cells of the table become the vertices of the Ferrers-graph, and two vertices are connected by an edge if and only if their cells violate the condition of a Ferrers-relation. Here, violating the condition of a Ferrers-relation means that the two 'partner' cells – which together with the two empty cells define the corners of a rectangle – both contain a cross. Hence, in the small example

|   |   |
|---|---|
| × |   |
|   | × |

the two empty cells are vertices of the corresponding Ferrers-graph, and they are connected by an edge:

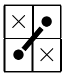
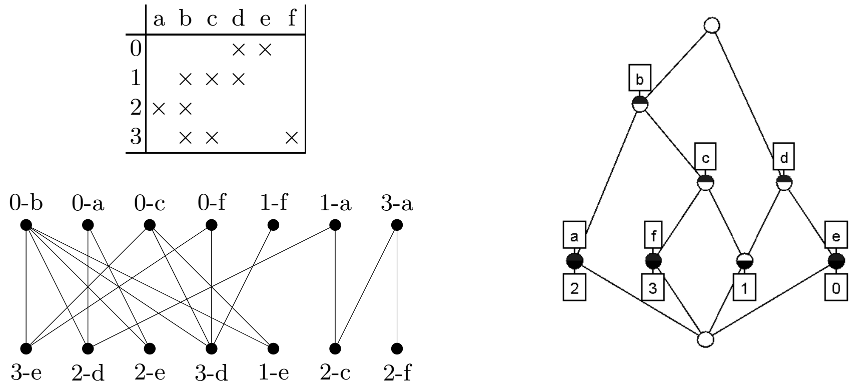


Figure 11 demonstrates by the example of two edges how the Ferrers-graph of a formal context is constructed. In the left part of the figure, the two vertices $(2, c)$ and $(3, a)$ of the Ferrers-graph have to be connected by an edge since their partner cells $(2, a)$ and $(3, c)$ bear crosses. The right part of the figure demonstrates that the vertices $(3, b)$ and $(0, e)$ have to be connected by an edge too. As an example for two non-connected vertices of the Ferrers-graph consider the vertices $(2, c)$ and $(3, d)$. They are not connected by an edge in the Ferrers-graph since their partner cell $(2, d)$ does not bear a cross.

The whole Ferrers graph for this example context is given in Fig. 12. Here, the edges of the graph are labeled by the cells of the cross table of the formal context. Note that the Ferrers-graph is bipartite which is in accordance with the Hasse-planarity of the corresponding concept lattice shown in Fig. 12. However, the example set of classes is not S-sortable since the node labeled $f$ does not lie on the S-graph of the concept lattice. Hence, by the main theorem on S-sortability it follows that the concept lattice of the enlarged set of classes is not Hasse-planar and that its Ferrers-graph is not bipartite. Both, the enlarged concept lattice and its corresponding Ferrers-graph are given in Fig. 13. As demonstrated by the edge between the vertices 2-f and 9-b, the Ferrers-graph is not bipartite.

The Ferrers-graph of a formal context is bipartite if it is possible to assign its vertices to two disjoint classes such that each edge connects vertices which belong to different classes. This property can easily be algorithmically verified



|   | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| 0 |   |   |   | × | × |   |
| 1 |   | × | × | × |   |   |
| 2 | × | × |   |   |   |   |
| 3 |   | × | × |   |   | × |

**Fig. 12.** example of a bipartite Ferrers-graph (upper left: formal context; lower left: Ferrers-graph; right: concept lattice)

by assigning an arbitrary start vertex to the first class and assigning all vertices which are connected with it to the second class. Every time a vertex is assigned to one class, all neighbor vertices are assigned to the other class. This procedure has to be repeated for every connected component of the Ferrers-graph. If at any point a vertex has to be assigned to both classes, the Ferrers-graph is necessarily not bipartite. But if it is possible to assign all vertices to the classes without conflicts, the Ferrers-graph is bipartite. Hence, condition 3 offers a possibility to check algorithmically whether a set of classes is S-sortable or not.

## 3 Identifying good candidates for duplication

The aim of this section is to illustrate how the three conditions for S-sortability can be applied in the construction of S-alphabets with duplications in the case of non S-sortable sets of classes. It turns out that for different sets of classes different strategies have to be chosen in order to tackle the problem of identifying those elements which have to be duplicated in order to get an S-alphabet with as few duplications as possible and a minimal number of markers.

First, it will be demonstrated by the examples in Fig. 12 and Fig. 13 how in some cases condition 3 can be applied in order to identify minimal S-alphabets in the case of non-sortable sets of classes. Let therefore

$$\mathcal{A} = \{a, b, c, d, e, f\} \quad \text{and} \quad \Phi = \{\{d, e\}, \{b, c, d\}, \{a, b\}, \{b, c, f\}\} .$$

As the Ferrers-graph in Fig. 13 of the enlarged set of classes is not bipartite, it is not possible to give an S-alphabet of $(\mathcal{A}, \Phi)$ without duplicated elements. The task is now to identify those elements whose duplication leads to an S-alphabet with as few duplicated elements as possible and a minimized marker set.

The Ferrers-graph in Fig. 13 indicates that the elements $b$ and $f$ cause the graph to be non-bipartite. Duplicating $f$ would be pointless since $f$ is an element of only one class, namely $\{b, c, f\}$. Therefore, it should be tried to duplicate $b$ such that the set of classes gets S-sortable and thus S-encodable.

|   | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| 0 |   |   |   | × | × |   |
| 1 |   | × | × | × |   |   |
| 2 | × | × |   |   |   |   |
| 3 |   | × | × |   |   | × |
| 4 | × |   |   |   |   |   |
| 5 |   | × |   |   |   |   |
| 6 |   |   | × |   |   |   |
| 7 |   |   |   | × |   |   |
| 8 |   |   |   |   | × |   |
| 9 |   |   |   |   |   | × |

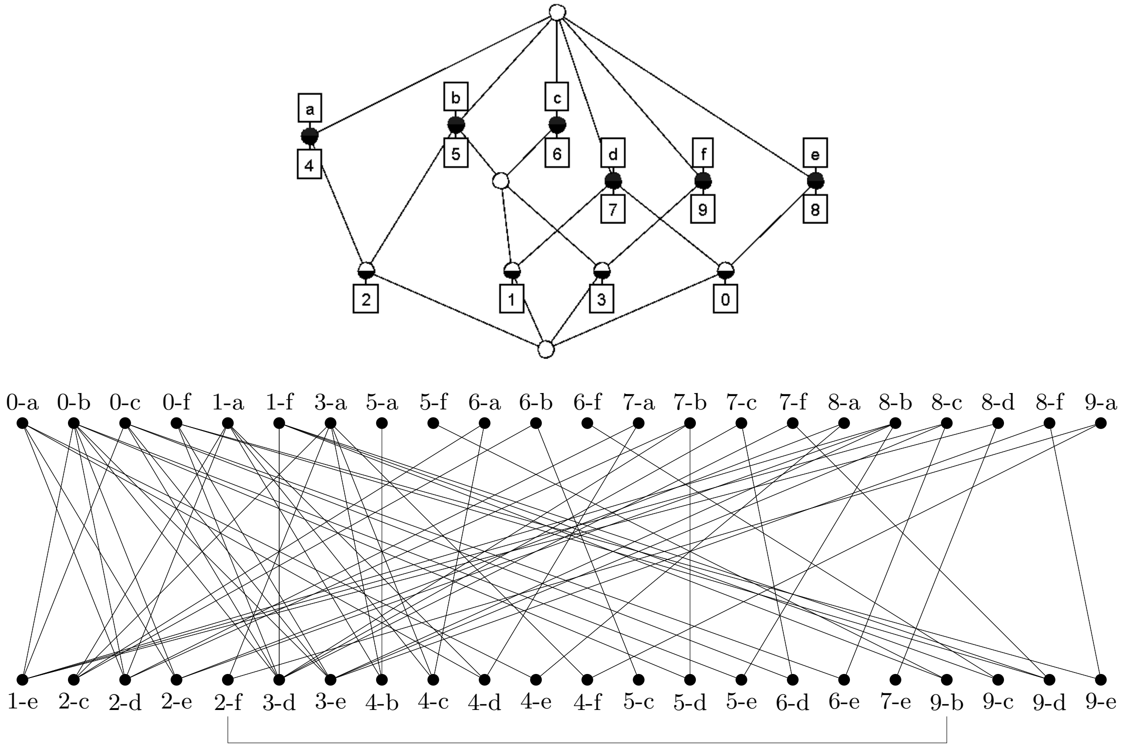

a
b
c
d
f
e
4
5
6
7
9
8
2
1
3
0
0-a 0-b 0-c 0-f 1-a 1-f 3-a 5-a 5-f 6-a 6-b 6-f 7-a 7-b 7-c 7-f 8-a 8-b 8-c 8-d 8-f 9-a
1-e 2-c 2-d 2-e 2-f 3-d 3-e 4-b 4-c 4-d 4-e 4-f 5-c 5-d 5-e 6-d 6-e 7-e 9-b 9-c 9-d 9-e

**Fig. 13.** example of a non-bipartite Ferrers-graph (left: formal context; upper right: concept lattice; lower right: Ferrers-graph

|   | a | c | d | e | f |
|---|---|---|---|---|---|
| 0 |   |   | × | × |   |
| 1 |   | × | × |   |   |
| 2 | × |   |   |   |   |
| 3 |   | × |   |   | × |

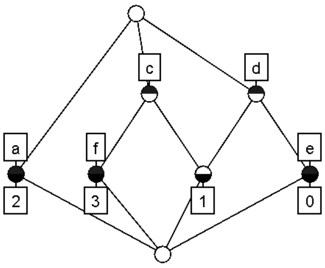


**Fig. 14.** formal context and concept lattice of the set of classes in Fig. 12 reduced by $b$

|   | a | b | b' | c | d | e | f |
|---|---|---|---|---|---|---|---|
| 0 |   |   |   |   | × | × |   |
| 1 |   |   | × | × | × |   |   |
| 2 | × | × |   |   |   |   |   |
| 3 |   | × |   | × |   |   | × |

|   | a | b | b' | c | d | e | f |
|---|---|---|---|---|---|---|---|
| 0 |   |   |   |   | × | × |   |
| 1 |   |   | × | × | × |   |   |
| 2 | × | × |   |   |   |   |   |
| 3 |   |   | × | × |   |   | × |

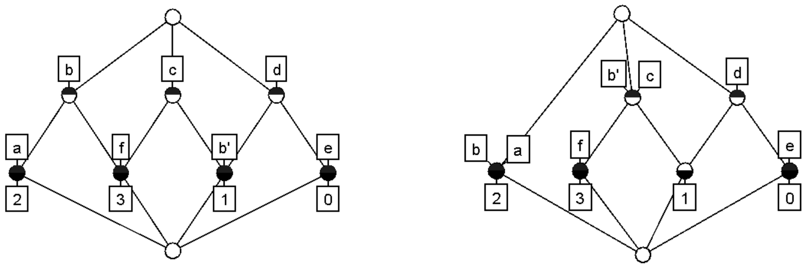


**Fig. 15.** formal contexts and concept lattices for possible duplications of $b$ (cf. Fig. 12 and Fig. 13)

First, by condition 2 Fig. 14 indicates that the set of classes becomes S-sortable if $b$ is completely removed. Furthermore, if a minimal S-alphabet can be gained by duplicating $b$, Fig. 14 restricts the order of the elements $a, c, d, e$ and $f$. On the basis of the S-graph of the concept lattice in Fig. 14 the following four minimal S-alphabets of the set of classes reduced by $b$ can be identified (the marker positions are indicated by vertical lines):

$$a|fc|d|e| \qquad fc|d|e|a| \qquad a|ed|c|f| \qquad ed|c|f|a| \; .$$

Adding a copy of $b$ such that the resulting set of classes becomes S-sortable leads to one of the two formal contexts and corresponding concept lattices in Fig. 15. From both concept lattices one can read off S-alphabets with a minimum of four marker elements. As four markers are already needed in an S-alphabet of the set of classes reduced by $b$, all S-alphabets with minimal marker sets which can be read off the S-graphs of the two concept lattices in Fig. 15 are minimal (e.g., $ab|fc|b'd|e|$, $ed|b'c|fb|a$, $ab|fb'c|d|e|$, $ed|b'c|f|ab|$, $\ldots$).

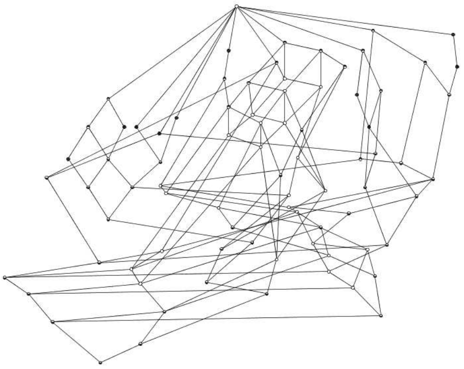


**Fig. 16.** concept lattice of Pāṇini's pratyāhāra-context

|  | $h$ | $l$ | $v$ |
|---|---|---|---|
| $\{h,l\}$ | × | × |  |
| $\{h,v\}$ | × |  | × |
| $\{v,l\}$ |  | × | × |

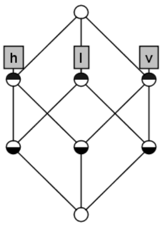
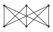


**Fig. 17.** formal context of three independent elements and its concept lattice

The analysis of Ferrers-graphs is not always as informative as in the case of the discussed example, where one single edge can be identified that destroys the bipartity of the graph. Therefore, another example that involves a different method for the identification of good candidates for duplication will be presented: Let the set of classes consists of those sound classes used in Pāṇini's Sanskrit grammar that are denoted by pratyāhāras. The concept lattice of the corresponding pratyāhāra-context is given in Fig. 16. As mentioned before, it is not Hasse-planar, but proving that a graph like the one in Fig. 16 is not planar may be hard. In **?** the proof is based on the *criterion of Kuratowski*, which states that a graph is planar, i.e. drawable without intersecting edges, if and only if it contains neither the graph nor the graph  as a minor.[3] In **?** it is shown that the graph in Fig. 16 has as a minor by identifying a fitting section of the graph.

Instead of applying Kuratowski's criterion directly it is easier to work with a derived necessary condition for S-sortability: Figure 17 shows a context of three independent elements and its concept lattice. An ordered set like this concept lattice is Hasse-planar if and only if the graph which is the result of adding

[3] A graph is the *minor* of an other graph if it can be constructed from the latter by removing vertices and edges and contracting some of the remaining edges.

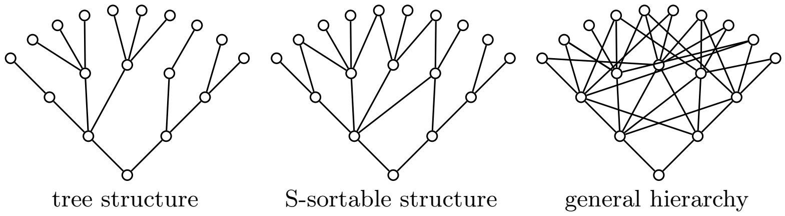
an extra edge connecting the bottom and the top node in its Hasse-diagram is planar. It can be easily verified that in the case of a concept lattice of three independent elements the resulting graph has as a minor. It follows that whenever a set of classes has three independent elements it is not S-sortable. Three elements are said to be independent if for any pair of them there exists a set in the concept lattice which contains both, but not the third. The set of all independent triples of a formal context can be extracted algorithmically.

In the case of Pāṇini's pratyāhāra-context, one finds 249 independent triples. Interestingly, all of them include the sound *h* and no other sound element is included in all of them. Hence, *h* is the best candidate for duplication as *h* offers the only possibility to destroy all independent triples by a single duplication and thus to order the sounds in an S-alphabet with just one duplicated element. By an analysis of the concept lattice of the pratyāhāra-context reduced by *h*, it has been proven in **?** that in the Śivasūtras Pāṇini has chosen a way of duplicating *h* that leads to a minimal S-alphabet.

## 4 Conclusion

The analyses of the various examples in this paper demonstrate how the three sufficient conditions for S-sortability offer different approaches for the construction of minimal S-alphabets which interlock and complement one another. As the problem of constructing minimal S-alphabets inherently bears the danger of combinatoric explosion, it is important to check which solution strategy is the most efficient in each individual case. One can benefit from the fact that the three conditions of S-sortability support different ways of tackling the problem. For instance, whether a graph is bipartite can be checked algorithmically while the question whether all labels lie on the S-graph of a concept lattice can be answered by simply looking at it. Hence, S-alphabets should be constructed semi-automatically by considering the application of all presented strategies.

In fact, deciding whether Pāṇini has actually chosen an optimal way of arranging the sounds in the Śivasūtras is more intricate than presented here. We have simplified the problem to the problem of constructing a minimal S-alphabet to the set of sound classes which are denoted by pratyāhāras in Pāṇini's grammar. But due to the following reasons this is not the exact problem which Pāṇini faced: First of all, not all sound classes in Pāṇini's grammar are denoted by pratyāhāras. For instance, Pāṇini also makes use of the older *varga*-classification of sounds, or sometimes he even simply lists the sounds involved in a rule. Second, Pāṇini permits overgeneralized rules by using a pratyāhāra in a rule that denotes a larger class of sounds than the one to which the rule actually applies (cf. **?**). Third, the order of the sounds in the Śivasūtras does not only depend on the classes which need to be encoded by pratyāhāras. A phonological rule, which claims that sounds of class $A$ are replaced by sounds of class $B$, also has to ensure that a replaced element of class $A$ is replaced by its counterpart of class $B$. In Pāṇini's grammar, a special meta-rule guarantees that the sounds are replaced by their counterparts according to their position in the sound lists



tree structure S-sortable structure general hierarchy

**Fig. 18.** mid-position of S-sortable structures

denoted by the pratyāhāras (cf. footnote 1). Hence, a deeper analysis of the use of the sound classes in Pāṇini's grammar is still necessary in order to decide whether the Śivasūtras are optimal.

We will conclude by some remarks on the promising prospect of revitalizing Pāṇini's Śivasūtra-technique in order to approach some modern problems. A first field of problems is tackled in **?**, namely the problem that quite often one is forced to order things linearly although they could be more naturally organized in a non-linear hierarchy (e.g., books on bookshelves, clothes on racks, ...). S-orders may offer a way out as they order elements linearly, but in a sense bundle elements of one class up by keeping the distances between them small.

Another possible, but yet unexplored application area of the presented formalization of Pāṇini's technique is data representation in Computer Science. Data structures in Computer Science are encoded linearly as classical programming languages are inherently linear. Since tree structures can be encoded as nested lists, many formalisms only allow for tree structures and leave polyhierarchies out. However, in knowledge engineering multiple inheritance relations are central and thus polyhierarchies are badly needed. In this dilemma, S-sortable sets of classes could take over the position of tree structures due to the fact that they can be encoded linearly by lists with indexed brackets. Furthermore, they build up a hierarchical structure which takes a mid-position between tree structures and general hierarchical structures (cf. Fig. 18): They can be represented in a plane drawing like tree structures, but allow, at least in a limited way, for multiple inheritance like general polyhierarchies. A promising task is to explore to what extent Pāṇini's Śivasūtra-technique can be employed for the representation of hierarchies in order to allow at least for limited multiple inheritance without loosing the advantages of an efficient linear encoding and processing of hierarchical relations. The idea is to extend, for some tasks, the class of admissible hierarchies from tree-shaped hierarchies to S-sortable ones. More on this idea can be found in **?**.

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